

SM015/2

PSPM 1

2019/2020



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KOLEJ MATRIKULASI KEDAH

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Questions

SECTION A [45 marks]

This section consists of 5 questions. Answer all questions.

1. Polynomial $P(x) = 12x^3 - px^2 - qx + 8$ is divisible by $3x^2 - 7x + 4$. Find the values of p and q . Hence, factorise $P(x)$ completely.

[8 marks]

2. a) Show that $\operatorname{cosec} \theta - \cot \theta$ can be simplified as $\tan \frac{\theta}{2}$.

[5 marks]

- b) Hence, find the values of $\tan 15^\circ$ and $\sec^2 15^\circ$. Give your answer in the form $a + b\sqrt{c}$, where a , b and c are integers.

[5 marks]

3. a) A function $f(x)$ is defined as

$$f(x) = \begin{cases} \frac{9 - x^2}{x - 3}, & x < 3 \\ -6, & x = 3 \\ \frac{162x - 54x^2}{x^3 - 27}, & x > 3 \end{cases}$$

Determine whether $f(x)$ is continuous at $x = 3$.

[7 mark]

- b) Find $\lim_{x \rightarrow \infty} \frac{5x^3 + 4x - 9}{7 - 3x^3}$.

[3 mark]

4. a) Given $y = x^2 e^{-3x}$, find $\frac{d^2y}{dx^2}$. Give your answer in the simplest form.

[5 mark]

- b) Given $x^2 y = \sin(2x^2 + \pi)$.

Show that $2y(1 + 8x^4) + 4x \frac{dy}{dx} + x^2 \frac{d^2y}{dx^2} = 4 \cos(2x^2 + \pi)$.

[7 marks]



5. Given $f(x) = x^2 - \frac{6}{x}$, where $x \neq 0$. Find the coordinate(s) of stationary point and determine whether it is a relative maximum or a relative minimum. Give the coordinates correct to 3 decimal places.

[5 marks]

SECTION B [25 marks]*This section consists of 2 question. Answer all questions.*

1. The function $g(x)$ is defined by

$$g(x) = \begin{cases} m^2 - 8, & x \leq 2 \\ \frac{x-2}{\sqrt{2x}-2}, & 2 < x \leq 8 \\ \frac{|8-x|}{x-8} + k, & x > 8 \end{cases}$$

Where m and k is constant.

- a. If $\lim_{x \rightarrow 2} g(x)$ exists, find the values of m .
- b. Find the values of k such that $g(x)$ is discontinuous at $x = 8$.
2. Given a right circular cylinder with height $2h$ and radius $r = \sqrt{a^2 - h^2}$, which is inscribed in a sphere of fixed radius a .
- a. Show that the volume of the cylinder is $V = 2\pi(a^2 - h^2)h$.
- b. From part 2(a), find the maximum value of the volume, V in terms of π as the height, h varies. Hence, state the value of the volume if $a = 3$.
- c. Show that the ratio of the volume of the sphere to the maximum volume of the cylinder is $\sqrt{3}:1$.

[5 marks]

[3 marks]

[5 marks]

[9 marks]

[3 marks]

END OF QUESTION PAPER

Question A1

1. Polynomial $P(x) = 12x^3 - px^2 - qx + 8$ is divisible by $3x^2 - 7x + 4$. Find the values of p and q . Hence, factorise $P(x)$ completely.

[8 marks]

SOLUTION

$$P(x) = 12x^3 - px^2 - qx + 8$$

$$D(x) = 3x^2 - 7x + 4 = (3x - 4)(x - 1)$$

$$P(1) = 0$$

$$P\left(\frac{4}{3}\right) = 0$$

$$P(1) = 12(1)^3 - p(1)^2 - q(1) + 8 = 0$$

$$p + q = 20$$

$$p = 20 - q \quad \dots\dots\dots (1)$$

$$P\left(\frac{4}{3}\right) = 12\left(\frac{4}{3}\right)^3 - p\left(\frac{4}{3}\right)^2 - q\left(\frac{4}{3}\right) + 8 = 0$$

$$\frac{768}{27} - \frac{16}{9}p - \frac{4}{3}q + 8 = 0$$

$$768 - 48p - 36q + 216 = 0$$

$$48p + 36q = 984 \quad \dots\dots\dots (2)$$

Substitute (1) into (2)

$$48(20 - q) + 36q = 984$$

$$960 - 48q + 36q = 984$$

$$-12q = 24$$

$$q = -2$$

$$p = 22$$

$$P(x) = 12x^3 - 22x^2 + 2x + 8$$



$$\begin{array}{r}
 3x^2 - 7x + 4 \overline{) 12x^3 - 22x^2 + 2x + 8} \\
 \underline{12x^3 - 28x^2 + 16x} \\
 6x^2 - 14x + 8 \\
 \underline{6x^2 - 14x + 8} \\
 0
 \end{array}$$

$$\begin{aligned}
 P(x) &= 12x^3 - 22x^2 + 2x + 8 \\
 &= (3x^2 - 7x + 4)(4x + 2) \\
 &= (3x - 4)(x - 1)(4x + 2) \\
 &= 2(3x - 4)(x - 1)(2x + 1)
 \end{aligned}$$

Alternative Method:

$$P(x) = 12x^3 - px^2 - qx + 8$$

$$D(x) = 3x^2 - 7x + 4 = (3x - 4)(x - 1)$$

$$P(1) = 0 \rightarrow (x - 1) \text{ is a factor of } P(x)$$

$$P\left(\frac{4}{3}\right) = 0 \rightarrow (3x - 4) \text{ is a factor of } P(x)$$

$$P(x) = Q(x)D(x)$$

$$12x^3 - px^2 - qx + 8 = (3x - 4)(x - 1)(ax + b)$$

$$12x^3 - px^2 - qx + 8 = (3x^2 - 3x - 4x + 4)(ax + b)$$

$$12x^3 - px^2 - qx + 8 = (3x^2 - 7x + 4)(ax + b)$$

$$12x^3 - px^2 - qx + 8 = 3ax^3 + 3bx^2 - 7ax^2 - 7bx + 4ax + 4b$$

$$12x^3 - px^2 - qx + 8 = 3ax^3 + (3b - 7a)x^2 + (4a - 7b)x + 4b$$

Compare the coefficient:

$$x^3: \quad 3a = 12$$

$$a = 4$$



$$x^0: \quad 8 = 4b$$

$$b = 2$$

$$x^2: \quad -p = 3b - 7a$$

$$-p = 3(2) - 7(4)$$

$$p = 22$$

$$x: \quad -q = 4a - 7b$$

$$-q = 4(4) - 7(2)$$

$$q = -2$$

$$p = 22; q = -2$$

$$P(x) = (3x - 4)(x - 1)(4x + 2)$$

$$= 2(3x - 4)(x - 1)(2x + 1)$$



Question A2

2. a) Show that $\operatorname{cosec} \theta - \cot \theta$ can be simplified as $\tan \frac{\theta}{2}$.

[5 marks]

b) Hence, find the values of $\tan 15^\circ$ and $\sec^2 15^\circ$. Give your answer in the form of $a + b\sqrt{c}$, where a, b and c are integers.

[5 marks]

SOLUTION

$$\begin{aligned}
 \text{a) } \operatorname{cosec} \theta - \cot \theta &= \frac{1}{\sin \theta} - \frac{\cos \theta}{\sin \theta} \\
 &= \frac{1 - \cos \theta}{\sin \theta} \\
 &= \frac{1 - (1 - 2\sin^2 \frac{\theta}{2})}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \\
 &= \frac{2\sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \\
 &= \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \\
 &= \tan \frac{\theta}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{b) } \tan \frac{\theta}{2} &= \operatorname{cosec} \theta - \cot \theta \\
 \tan 15^\circ &= \tan \left(\frac{30^\circ}{2} \right) \\
 &= \operatorname{cosec} 30^\circ - \cot 30^\circ \\
 &= \frac{1}{\sin 30^\circ} - \frac{\cos 30^\circ}{\sin 30^\circ} \\
 &= \frac{1}{\frac{1}{2}} - \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \\
 &= 2 - \sqrt{3}
 \end{aligned}$$



$$\sec^2 \theta = \tan^2 \theta + 1$$

$$\sec^2 15^\circ = \tan^2 15^\circ + 1$$

$$= (\tan 15^\circ)^2 + 1$$

$$= (2 - \sqrt{3})^2 + 1$$

$$= (4 + 3 - 4\sqrt{3}) + 1$$

$$= 8 - 4\sqrt{3}$$

$$= 4(2 - \sqrt{3})$$



Question A3

3. a) A function $f(x)$ is defined as

$$f(x) = \begin{cases} \frac{9-x^2}{x-3}, & x < 3 \\ -6, & x = 3 \\ \frac{162x-54x^2}{x^3-27}, & x > 3 \end{cases}$$

Determine whether $f(x)$ is continuous at $x = 3$.

[7 mark]

b) Find $\lim_{x \rightarrow \infty} \frac{5x^3+4x-9}{7-3x^3}$.

[3 mark]

SOLUTION

$$\text{a) } f(x) = \begin{cases} \frac{9-x^2}{x-3}, & x < 3 \\ -6, & x = 3 \\ \frac{162x-54x^2}{x^3-27}, & x > 3 \end{cases}$$

At $x = 3$

$$f(3) = -6$$

$$\begin{aligned} \lim_{x \rightarrow 3^-} f(x) &= \lim_{x \rightarrow 3^-} \left(\frac{9-x^2}{x-3} \right) \\ &= \lim_{x \rightarrow 3^-} \left[\frac{(3+x)(3-x)}{x-3} \right] \\ &= \lim_{x \rightarrow 3^-} \left[\frac{-(3+x)(x-3)}{x-3} \right] \\ &= \lim_{x \rightarrow 3^-} [-(3+x)] \\ &= -(3+3) \\ &= -6 \end{aligned}$$



$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} \left(\frac{162x - 54x^2}{x^3 - 27} \right)$$

$$= \lim_{x \rightarrow 3^+} \left[\frac{-54x(x-3)}{(x-3)(x^2+3x+9)} \right]$$

$$= \lim_{x \rightarrow 3^+} \left[\frac{-54x}{x^2+3x+9} \right]$$

$$= \frac{-54(3)}{3^2+3(3)+9}$$

$$= \frac{-162}{27}$$

$$= -6$$

$$\begin{array}{r} x-3 \overline{) x^3 - 27} \\ \underline{x^3 - 3x^2} \\ 3x^2 - 27 \\ \underline{3x^2 - 9x} \\ 9x - 27 \\ \underline{9x - 27} \\ 0 \end{array}$$

$$\lim_{x \rightarrow 3} f(x) = -6$$

Since $f(3) = \lim_{x \rightarrow 3} f(x) = -6$, therefore

$f(x)$ is continuous at $x = 3$

$$\text{b) } \lim_{x \rightarrow \infty} \frac{5x^3 + 4x - 9}{7 - 3x^3} = \lim_{x \rightarrow \infty} \frac{5x^3 + 4x - 9}{7 - 3x^3}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{5x^3 + 4x - 9}{x^3}}{\frac{7 - 3x^3}{x^3}}$$

$$= \lim_{x \rightarrow \infty} \frac{5 + \frac{4}{x^2} - \frac{9}{x^3}}{\frac{7}{x^3} - 3}$$

$$= \frac{5 + 0 - 0}{0 - 3}$$

$$= -\frac{5}{3}$$



Question A4

4. a) Given $y = x^2 e^{-3x}$, find $\frac{d^2y}{dx^2}$. Give your answer in the simplest form.

[5 mark]

b) Given $x^2 y = \sin(2x^2 + \pi)$.

Show that $2y(1 + 8x^4) + 4x \frac{dy}{dx} + x^2 \frac{d^2y}{dx^2} = 4 \cos(2x^2 + \pi)$.

[7 marks]

SOLUTION

$$a) y = x^2 e^{-3x}$$

$$u = x^2$$

$$v = e^{-3x}$$

$$u' = 2x$$

$$v' = -3e^{-3x}$$

$$\frac{dy}{dx} = uv' + vu'$$

$$= (x^2)(-3e^{-3x}) + (e^{-3x})(2x)$$

$$= e^{-3x}[-3x^2 + 2x]$$

$$u = -3x^2 + 2x$$

$$v = e^{-3x}$$

$$u' = -6x + 2$$

$$v' = -3e^{-3x}$$

$$\frac{d^2y}{dx^2} = (-3x^2 + 2x)(-3e^{-3x}) + (e^{-3x})(-6x + 2)$$

$$= e^{-3x}[(-3)(-3x^2 + 2x) + (-6x + 2)]$$

$$= e^{-3x}[9x^2 - 6x - 6x + 2]$$

$$= e^{-3x}[9x^2 - 12x + 2]$$

$$b) x^2 y = \sin(2x^2 + \pi)$$

$$x^2 \frac{dy}{dx} + 2xy = 4x \cos(2x^2 + \pi)$$



$$\left[x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} \right] + \left[2x \frac{dy}{dx} + 2y \right] = [-4x(4x) \sin(2x^2 + \pi) + 4\cos(2x^2 + \pi)]$$

$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + 2x \frac{dy}{dx} + 2y = -16x^2 \sin(2x^2 + \pi) + 4\cos(2x^2 + \pi)$$

$$x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = -16x^2 (x^2y) + 4\cos(2x^2 + \pi)$$

$$x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y + 16x^2 (x^2y) = 4\cos(2x^2 + \pi)$$

$$x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y(1 + 8x^4) = 4\cos(2x^2 + \pi)$$

$$2y(1 + 8x^4) + 4x \frac{dy}{dx} + x^2 \frac{d^2y}{dx^2} = 4 \cos(2x^2 + \pi)$$

Given:

$$x^2y = \sin(2x^2 + \pi)$$



Question A5

5. Given $f(x) = x^2 - \frac{6}{x}$, where $x \neq 0$. Find the coordinate(s) of stationary point and determine whether it is a relative maximum or a relative minimum. Give the coordinates correct to 3 decimal places.

[5 marks]

SOLUTION

$$\begin{aligned} f(x) &= x^2 - \frac{6}{x} \\ &= x^2 - 6x^{-1} \end{aligned}$$

$$\begin{aligned} f'(x) &= 2x + 6x^{-2} \\ &= 2x + \frac{6}{x^2} \end{aligned}$$

Let $f'(x) = 0$

$$2x + \frac{6}{x^2} = 0$$

$$2x^3 + 6 = 0$$

$$2x^3 = -6$$

$$x^3 = -3.$$

$$x = -1.442$$

When $x = -1.442$,

$$\begin{aligned} f(x) &= (-1.442)^2 - \frac{6}{-1.442} \\ &= 6.241 \end{aligned}$$

The stationary point = $(-1.442, 6.241)$

$$\begin{aligned} \frac{d^2y}{dx^2} &= 2 - 12x^{-3} \\ &= 2 - \frac{12}{x^3} \end{aligned}$$

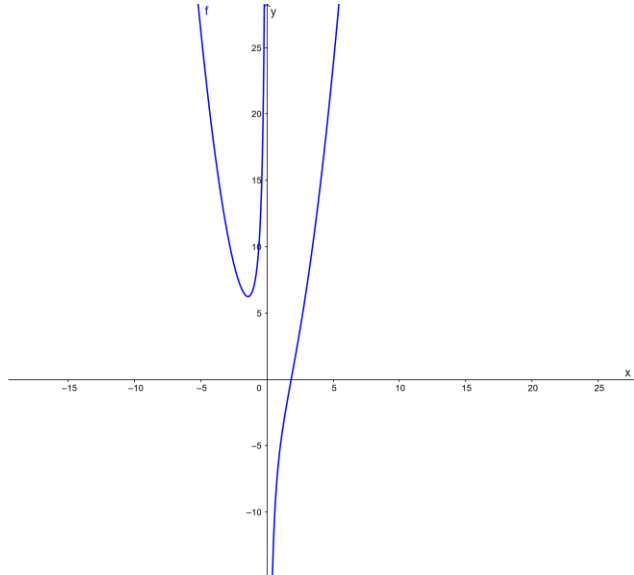
When $x = -1.44$

$$\frac{d^2y}{dx^2} = 2 - \frac{12}{(-1.44)^3} = 6.002 > 0 \text{ (Minimum)}$$

\therefore The point $(-1.44, 6.241)$ is relative minimum.



Graph: (Only for illustration purpose)



Question B1

1. The function $g(x)$ is defined by

$$g(x) = \begin{cases} m^2 - 8, & x \leq 2 \\ \frac{x-2}{\sqrt{2x-2}}, & 2 < x \leq 8 \\ \frac{|8-x|}{x-8} + k, & x > 8 \end{cases}$$

Where m and k is constant.

a. If $\lim_{x \rightarrow 2} g(x)$ exists, find the values of m .

[5 marks]

b. Find the values of k such that $g(x)$ is discontinuous at $x = 8$.

[3 marks]

SOLUTION

$$g(x) = \begin{cases} m^2 - 8, & x \leq 2 \\ \frac{x-2}{\sqrt{2x-2}}, & 2 < x \leq 8 \\ \frac{|8-x|}{x-8} + k, & x > 8 \end{cases}$$

a) $\lim_{x \rightarrow 2} g(x)$

$$\lim_{x \rightarrow 2^-} g(x) = \lim_{x \rightarrow 2^+} g(x)$$

$$\lim_{x \rightarrow 2^-} m^2 - 8 = \lim_{x \rightarrow 2^+} \frac{x-2}{\sqrt{2x-2}}$$

$$m^2 - 8 = \lim_{x \rightarrow 2^+} \frac{x-2}{\sqrt{2x-2}} \frac{\sqrt{2x+2}}{\sqrt{2x+2}}$$

$$m^2 - 8 = \lim_{x \rightarrow 2^+} \frac{(x-2)(\sqrt{2x+2})}{2x-4}$$

$$m^2 - 8 = \lim_{x \rightarrow 2^+} \frac{(x-2)(\sqrt{2x+2})}{2(x-2)}$$

$$m^2 - 8 = \lim_{x \rightarrow 2^+} \frac{\sqrt{2x+2}}{2}$$



$$m^2 - 8 = \frac{\sqrt{2(2)+2}}{2}$$

$$m^2 - 8 = 2$$

$$m^2 = 10$$

$$m = \pm\sqrt{10}$$

b) $g(x)$ is discontinuous at $x = 8$

$$f(8) = \frac{8-2}{\sqrt{2(8)}-2}$$

$$f(8) = \frac{6}{2} = 3$$

$$\lim_{x \rightarrow 8^+} \frac{|8-x|}{x-8} + k = \lim_{x \rightarrow 8^+} \frac{-8+x}{x-8} + k$$

$$= \lim_{x \rightarrow 8^+} 1 + k$$

$$= 1 + k$$

Since $g(x)$ is discontinuous at $x = 8$

$$1 + k \neq 3$$

$$k \neq 2$$

$$k \in \mathbb{R} \setminus \{2\}$$

$$|8-x| = \begin{cases} 8-x, & 8-x \geq 0 \\ -(8-x), & 8-x < 0 \end{cases}$$

$$= \begin{cases} 8-x, & x \leq 8 \\ -8+x, & x > 8 \end{cases}$$



Question B2

2. Given a right circular cylinder with height $2h$ and radius $r = \sqrt{a^2 - h^2}$, which is inscribed in a sphere of fixed radius a .

a. Show that the volume of the cylinder is $V = 2\pi(a^2 - h^2)h$.

[5 marks]

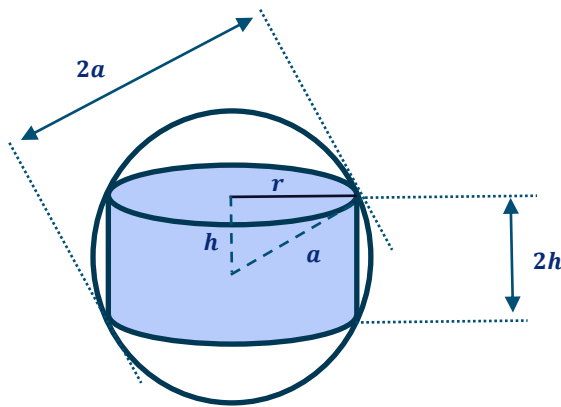
b. From part 2(a), find the maximum value of the volume, V in terms of π as the height, h varies. Hence, state the value of the volume if $a = 3$.

[9 marks]

c. Show that the ratio of the volume of the sphere to the maximum volume of the cylinder is $\sqrt{3}:1$.

[3 marks]

SOLUTION



$$\text{a) } r^2 + h^2 = a^2$$

$$r^2 = a^2 - h^2$$

$$r = \sqrt{a^2 - h^2}$$

$$\begin{aligned} \text{Volume of cylinder} &= \pi r^2 (2h) \\ &= 2\pi (\sqrt{a^2 - h^2})^2 h \\ &= 2\pi (a^2 - h^2) h \end{aligned}$$



$$b) V = 2\pi(a^2 - h^2)h$$

$$= 2\pi a^2 h - 2\pi h^3$$

$$\frac{dV}{dh} = 2\pi a^2 - 6\pi h^2$$

$$\text{Let } \frac{dV}{dh} = 0;$$

$$2\pi a^2 - 6\pi h^2 = 0$$

$$2\pi(a^2 - 3h^2) = 0$$

$$a^2 - 3h^2 = 0$$

$$3h^2 = a^2$$

$$h^2 = \frac{a^2}{3}$$

$$\frac{dV}{dh} = 2\pi a^2 - 6\pi h^2$$

$$\frac{d^2V}{dh^2} = -12\pi h < 0 \quad (\text{as } h > 0)$$

When $a = 3$

$$h^2 = \frac{3^2}{3} = 3$$

$$h = \sqrt{3}$$

$$r^2 = a^2 - h^2$$

$$= 3^2 - 3$$

$$= 6$$

$$V_{\max} = 2\pi(a^2 - h^2)h$$

$$= 2\pi(3h^2 - h^2)h$$

$$= 2\pi(2h^3)$$

$$= 4\pi h^3$$

$$= 4\pi\sqrt{3}^3$$

$$= 12\sqrt{3}\pi$$



c) Volume of sphere $V = \frac{4}{3}\pi r^3$

$$V_{max} = 4\pi h^3$$

$$V_S = \frac{4}{3}\pi(\sqrt{3h^2})^3 = \frac{4}{3}\pi(3\sqrt{3}h^3) = 4\sqrt{3}\pi h^3$$

$$\frac{V_S}{V_{max}} = \frac{4\sqrt{3}\pi h^3}{4\pi h^3}$$
$$= \frac{\sqrt{3}}{1}$$

$$V_S : V_{max} = \sqrt{3} : 1$$

