SM015/1
MATHEMATICS

2018/2019

Matriculation Programme Examination

1. a) Solve $\sqrt{6x+1} - \sqrt{x} = 3$.

   b) Determine the solution set of x which satisfies the inequality.

$$\frac{2}{x+1} < \frac{x}{x+3}$$

2. a) Find the sum of all integers from 5 to 950 which are divisible by 3.

   b) Expand $3(1+x)^{\frac{1}{4}}$ in ascending powers of x up to the fourth term. Hence, approximate $\sqrt[4]{80}$ correct to four decimal places.

3. Find the gradient of the curve $\cos(4xy) = \tan(xy^2) - 3y$ at the point where $x = 0$.

4. The parametric equations of a curve are $x = t + \frac{2}{t}$ and $y = 2t - \frac{4}{t}$, where $t \neq 0$. Show that $\frac{dy}{dx} = 2 + \frac{8}{t^2-2}$. Hence, find $\frac{d^2y}{dx^2}$ in term of t.

5. Water is poured into a right inverted cone of height h with a semi-vertical angle of $60°$ at a constant rate of $25\pi cm^3$ per second.

   a. Show that the rate of change of the height of water is $\frac{dh}{dt} = \frac{25}{3h^2}$.

   b. Find the rate of change of the height of water after 5 seconds.

   c. Given the height of the cone is 20cm, find the time taken to fill the cone completely with water.

END OF QUESTION PAPER

1. a) Solve $\sqrt{6x+1}-\sqrt{x}=3$.

   b) Determine the solution set of x which satisfies the inequality.

$$\frac{2}{x+1}<\frac{x}{x+3}$$

SOLUTION

1a)

$$\sqrt{6x+1}-\sqrt{x}=3$$

$$\left(\sqrt{6x+1}-\sqrt{x}\right)^2=3^2$$

$$(6x+1)+(x)-2\left(\sqrt{6x+1}\right)\left(\sqrt{x}\right)=9$$

$$7x+1-2\left(\sqrt{6x+1}\right)\left(\sqrt{x}\right)=9$$

$$7x-8=2\left(\sqrt{6x+1}\right)\left(\sqrt{x}\right)$$

$$(7x-8)^2=\left[2\left(\sqrt{6x+1}\right)\left(\sqrt{x}\right)\right]^2$$

$$49x^2+64-112x=4(6x+1)(x)$$

$$49x^2+64-112x=24x^2+4x$$

$$25x^2-116x+64=0$$

$$(25x-16)(x-4)=0$$

$$x=\frac{16}{25}\quad,\ x=4$$

Check:

$\sqrt{6x+1}-\sqrt{x}=3$	$\sqrt{6x+1}-\sqrt{x}=3$
When $x=\frac{16}{25}$	When $x=4$

$\sqrt{6x+1}-\sqrt{x}=\sqrt{6\left(\frac{16}{25}\right)+1}-\sqrt{\frac{16}{25}}$ $=\sqrt{\frac{121}{25}}-\sqrt{\frac{16}{25}}$ $=\frac{11}{5}-\frac{4}{5}$ $=\frac{7}{5}\neq 3$	$\sqrt{6x+1}-\sqrt{x}=\sqrt{6(4)+1}-\sqrt{4}$ $=\sqrt{25}-2$ $=3$

$\therefore x=4$

1b)

$$\frac{2}{x+1}<\frac{x}{x+3}$$

$$\frac{2}{x+1}-\frac{x}{x+3}<0$$

$$\frac{2(x+3)-x(x+1)}{(x+1)(x+3)}<0$$

$$\frac{2x+6-x^2-x}{(x+1)(x+3)}<0$$

$$\frac{-x^2+x+6}{(x+1)(x+3)}<0$$

$$\frac{x^2-x-6}{(x+1)(x+3)}>0$$

$$\frac{(x+2)(x-3)}{(x+1)(x+3)}>0$$

Critical value:

$x = -3, -2, -1, 3$

	$(-\infty, -3)$	$(-3, -2)$	$(-2, -1)$	$(-1,3)$	$(3, \infty)$
$x + 2$	-	-	+	+	+
$x - 3$	-	-	-	-	+
$x + 1$	-	-	-	+	+
$x + 3$	-	+	+	+	+
$\dfrac{(x+2)(x-3)}{(x+1)(x+3)}$	$\oplus$	-	$\oplus$	-	$\oplus$

$Solution\ set: \{x: x < -3 \cup -2 < x < -1 \cup x > 3\}$

2. a) Find the sum of all integers from 5 to 950 which are divisible by 3.

b) Expand $3(1+x)^{\frac{1}{4}}$ in ascending powers of x up to the fourth term. Hence, approximate $\sqrt[4]{80}$ correct to four decimal places.

SOLUTION

2a)

Integers:

5, 6, 7, 8, 9, 10,...950

Integers which are divisible by 3:

6, 9, 12, 15, ... 945, 948

$a = 6, \quad d = 3$

$T_n = 948$

$a + (n-1)d = 948$

$6 + (n-1)3 = 948$

$6 + 3n - 3 = 948$

$3n = 945$

$n = 315$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{315} = \frac{315}{2}[2(6) + (315-1)3]$

$= 150255$

2b)

$$3(1+x)^{\frac{1}{4}} = 3\left[1 + \frac{\left(\frac{1}{4}\right)}{1!}(x) + \frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)}{2!}(x)^2 + \frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{3!}(x)^3\right]$$

$$= 3\left[1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{7}{128}x^3\right]$$

$$= 3 + \frac{3}{4}x - \frac{9}{32}x^2 + \frac{21}{128}x^3$$

$$|x| < 1$$

$$-1 < x < 1$$

$$3(1+x)^{\frac{1}{4}} = [3^4(1+x)]^{\frac{1}{4}}$$

$$= (81 + 81x)^{\frac{1}{4}}$$

$$\sqrt[4]{80} = 80^{\frac{1}{4}}$$

$$Let\ 81 + 81x = 80$$

$$x = -\frac{1}{81}$$

$$\sqrt[4]{80} = 3 + \frac{3}{4}\left(-\frac{1}{81}\right) - \frac{9}{32}\left(-\frac{1}{81}\right)^2 + \frac{21}{128}\left(-\frac{1}{81}\right)^3$$

$$= 3 - \frac{1}{108} - \frac{1}{23326} - \frac{7}{22674816}$$

$$= 2.9907$$

3. Find the gradient of the curve $\cos(4xy) = \tan(xy^2) - 3y$ at the point where $x = 0$.

SOLUTION

$$\cos(4xy) = \tan(xy^2) - 3y$$

$$-\sin(4xy)\frac{d}{dx}(4xy) = sec^2(xy^2)\frac{d}{dx}(xy^2) - 3\frac{dy}{dx}$$

$$-\sin(4xy)\left[4x\frac{dy}{dx} + 4y\right] = sec^2(xy^2)\left[2xy\frac{dy}{dx} + y^2\right] - 3\frac{dy}{dx}$$

$$-4x\sin(4xy)\frac{dy}{dx} - 4y\sin(4xy) = 2xysec^2(xy^2)\frac{dy}{dx} + y^2sec^2(xy^2) - 3\frac{dy}{dx}$$

$$3\frac{dy}{dx} - 4x\sin(4xy)\frac{dy}{dx} - 2xysec^2(xy^2)\frac{dy}{dx} = 4y\sin(4xy) + y^2sec^2(xy^2)$$

$$\frac{dy}{dx}[3 - 4x\sin(4xy) - 2xysec^2(xy^2)] = 4y\sin(4xy) + y^2sec^2(xy^2)$$

$$\frac{dy}{dx} = \frac{4y\sin(4xy) + y^2sec^2(xy^2)}{3 - 4x\sin(4xy) - 2xysec^2(xy^2)}$$

$When\ x = 0$

$$\cos(4xy) = \tan(xy^2) - 3y$$

$$\cos[4(0)y] = \tan[(0)y^2] - 3y$$

$$\cos 0 = \tan 0 - 3y$$

$$1 = 0 - 3y$$

$$y = -\frac{1}{3}$$

$When\ x = 0;\ \ y = -\frac{1}{3}$

$$\frac{dy}{dx} = \frac{4y\sin(4xy) + y^2sec^2(xy^2)}{3 - 4x\sin(4xy) - 2xysec^2(xy^2)}$$

$$\frac{dy}{dx}=\frac{4\left(-\frac{1}{3}\right)\sin\left[4(0)\left(-\frac{1}{3}\right)\right]+\left(-\frac{1}{3}\right)^2 sec^2\left[(0)\left(-\frac{1}{3}\right)^2\right]}{3-4(0)\sin\left[4(0)\left(-\frac{1}{3}\right)\right]-2(0)\left(-\frac{1}{3}\right)sec^2\left[(0)\left(-\frac{1}{3}\right)^2\right]}$$

$$=\frac{4\left(-\frac{1}{3}\right)\sin 0+\left(-\frac{1}{3}\right)^2 sec^2 0}{3}$$

$$=\frac{4\left(-\frac{1}{3}\right)\sin 0+\left(-\frac{1}{3}\right)^2\left(\frac{1}{cos^2 0}\right)}{3}$$

$$=\frac{1}{27}$$

4. The parametric equations of a curve are $x = t + \frac{2}{t}$ and $y = 2t - \frac{4}{t}$, where $t \neq 0$. Show that $\frac{dy}{dx} = 2 + \frac{8}{t^2-2}$. Hence, find $\frac{d^2y}{dx^2}$ in term of t.

SOLUTION

$$x = t + \frac{2}{t} = t + 2t^{-1}$$

$$\frac{dx}{dt} = 1 - 2t^{-2}$$

$$= 1 - \frac{2}{t^2}$$

$$= \frac{t^2 - 2}{t^2}$$

$$y = 2t - \frac{4}{t} = 2t - 4t^{-1}$$

$$\frac{dy}{dt} = 2 + 4t^{-2}$$

$$= 2 + \frac{4}{t^2}$$

$$= \frac{2t^2 + 4}{t^2}$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{dy}}{\mathrm{dt}} \cdot \frac{dt}{dx}$$

$$= \left(\frac{2t^2 + 4}{t^2}\right) \cdot \frac{1}{\left(\frac{t^2 - 2}{t^2}\right)}$$

$$= \left(\frac{2t^2 + 4}{t^2}\right) \cdot \frac{t^2}{t^2 - 2}$$

$$= \frac{2t^2 + 4}{t^2 - 2}$$

$$\begin{array}{r|l} & 2 \\ t^2 - 2 & 2t^2 + 4 \\ & 2t^2 - 4 \\ \hline & 8 \end{array}$$

$$\frac{dy}{dx} = 2 + \frac{8}{t^2 - 2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left[\frac{dy}{dx}\right] \cdot \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}[2 + 8(t^2 - 2)^{-1}] \cdot \left(\frac{t^2}{t^2 - 2}\right)$$

$$= \left[0 - 8(t^2 - 2)^{-2}\frac{d}{dt}(t^2 - 2)\right] \cdot \left(\frac{t^2}{t^2 - 2}\right)$$

$$= \left[\frac{8}{(t^2 - 2)^2}(2t)\right] \cdot \left(\frac{t^2}{t^2 - 2}\right)$$

$$= \left[\frac{16t}{(t^2 - 2)^2}\right] \cdot \left(\frac{t^2}{t^2 - 2}\right)$$

$$= \frac{16t^3}{(t^2 - 2)^3}$$

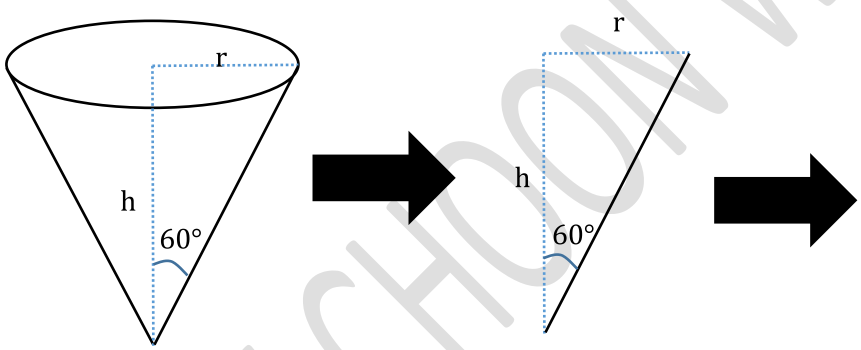
5. Water is poured into a right inverted cone of height h with a semi-vertical angle of 60° at a constant rate of $25\pi cm^3$ per second.

   a. Show that the rate of change of the height of water is $\frac{dh}{dt} = \frac{25}{3h^2}$.

   b. Find the rate of change of the height of water after 5 seconds.

   c. Given the height of the cone is 20cm, find the time taken to fill the cone completely with water.

SOLUTION



$$\tan 60° = \frac{r}{h}$$

$$\sqrt{3} = \frac{r}{h}$$

$$r = \sqrt{3}h$$

a)

$$\frac{\mathrm{dv}}{\mathrm{dt}} = 25\pi$$

$$\frac{dh}{dt} = \frac{dh}{dv} \cdot \frac{dv}{dt}$$

$$= \frac{dh}{dv}(25\pi)$$

$** \ We\ need\ to\ form\ an\ equation\ for\ \boldsymbol{v}\ in\ term\ of\ \boldsymbol{h}.$

$$v = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(\sqrt{3}h\right)^2 h$$

$$= \frac{1}{3}\pi(3h^2)h$$

$$= \pi h^3$$

$$\frac{dv}{dh} = 3\pi h^2$$

$$\frac{dh}{dt} = \left(\frac{dh}{dv}\right).\frac{dv}{dt}$$

$$= \left(\frac{1}{3\pi h^2}\right).(25\pi)$$

$$= \frac{25\pi}{3\pi h^2}$$

$$= \frac{25}{3h^2}$$

b) $when\ t = 5, find\ \frac{dh}{dt}$

$$\frac{\mathrm{dv}}{\mathrm{dt}} = 25\pi$$

$$v = \frac{dv}{dt}.t$$

$$\pi h^3 = (25\pi).(5)$$

$$h^3 = 125$$

$$h = 5$$

$$\frac{dh}{dt} = \frac{25}{3h^2}$$

$$= \frac{25}{3(5)^2}$$

$$= \frac{1}{3}cm/s$$

c) $When\ h = 20\ cm$

$$v = \pi h^3$$
$$= \pi(20)^3$$
$$= 8000\pi$$

$$v = \frac{dv}{dt}.t$$

$$8000\pi = (25\pi).t$$

$$t = 320s$$