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QS 015/2

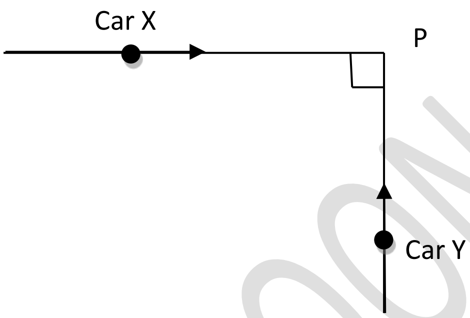
Matriculation Programme Examination

Semester I

Session 2014/2015

1. Given that $(x-2)$ is a factor of the polynomial $f(x) = ax^3 - 10x^2 + bx - 2$ where a and b are real numbers. If $f(x)$ is divided by $(x+1)$ the remainder is -24, find the values of a and b. Hence, find the remainder when $f(x)$ is divided by $(2x+1)$.
2. Solve the equation $2\ cos^2 x - 1 = \sin x\ for\ 0 \leq x \leq 2\pi$. Give your answer in terms of π.
3. Find the relative extremum of the curve $y = x^3 - 4x^2 + 4x$.
4. Car X is travelling east at a speed of 80km/h and car Y is travelling north at 100km/h as shown in the diagram below. Obtain an equation that describes the rate of change of the distance between the two cars.

   Hence, evaluate the rate of change of the distance between the two cars when car X is 0.15km and car Y is 0.08km from P.



5. Expand $(x+a)(x+b)^2$, a and b are real numbers with $b > 0$. Hence, find the values of a and b if $(x+a)(x+b)^2 = x^3 - 3x - 2$.

   Express $\frac{x^4-4x^2+5x-1}{x^3-3x-2}$ in the form of partial fractions.
6. (a) Express $\sin 6x - \sin 2x$ in product form. Hence, show that

   $$\sin 6x - \sin 2x + \sin 4x = 4\cos 3x \sin 2x \cos x.$$

   (b) Use the result in (a) to solve

   $$\sin 6x - \sin 2x + \sin 4x = \sin 2x \cos x \text{ for } 0 \leq x \leq 180°.$$
7. Find the limit of the following, if it exists.

   a. $\lim_{x\to -3} \frac{x+3}{x^3+27}$

   b. $\lim_{x\to -\infty} \frac{2x-1}{\sqrt{x^2-9}}$

   c. $\lim_{x\to 4} \frac{x^2-3x-4}{\sqrt{x}-2}$

8. Given that $f(x) = \begin{cases} 1+e^x & x \le 0 \\ \frac{x+6}{3-x} & 0 < x \le 4 \\ C & x > 4 \end{cases}$

Where C is a constant.

a) Determine whether $f(x)$ is continuous at $x = 0$.

b) Given that $f(x)$ is discontinuous at $x = 4$, determine the values of C.

c) Find the vertical asymptote of $f(x)$.

9. Consider the parametric equations of the curve

$x = cos^3\theta$ and $y = sin^3\theta,$ $0 < \theta \le 2\pi.$

a) Find $\frac{dy}{dx}$ and express your answer in terms of θ.

b) Find the value of $\frac{dy}{dx}$ if $x = \frac{\sqrt{2}}{4}$

c) Show that $\frac{d^2y}{dx^2} = \frac{1}{3cos^4\theta \sin\theta}$.

Hence, calculate $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{3}$.

10. (a) Use the first principle to find the derivative of $g(x) = \sqrt{1-x}$.

(b) Given that $e^y + xy + \ln(1+2x) = 1, x \ge 0$

Show that $(e^y + x)\frac{d^2y}{dx^2} + e^y\left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx} - \frac{4}{(1+2x)^2} = 0$.

Hence, find the value of $\frac{d^2y}{dx^2}$ at the point $(0,0)$.

END OF QUESTION PAPER

1. Given that $(x-2)$ is a factor of the polynomial $f(x) = ax^3 - 10x^2 + bx - 2$ where a and b are real numbers. If $f(x)$ is divided by $(x+1)$ the remainder is -24, find the values of a and b. Hence, find the remainder when $f(x)$ is divided by $(2x+1)$.

SOLUTION

$f(x) = ax^3 - 10x^2 + bx - 2$

$f(2) = 0$ — $(x-2)$ is a factor of $f(x)$

$a(2)^3 - 10(2)^2 + b(2) - 2 = 0$

$8a - 40 + 2b - 2 = 0$

$8a + 2b = 42$

$4a + b = 21$ .... (1)

$f(-1) = -24$ — $f(x)$ is divided by $(x+1)$ the remainder is -24

$a(-1)^3 - 10(-1)^2 + b(-1) - 2 = -24$

$-a - 10 - b - 2 = -24$

$a + b = 12$ .... (2)

$(1) - (2)$

$3a = 9$

$a = 3$

$b = 9$

$f(x) = 3x^3 - 10x^2 + 9x - 2$

$$f\left(-\frac{1}{2}\right) = 3\left(-\frac{1}{2}\right)^3 - 10\left(-\frac{1}{2}\right)^2 + 9\left(-\frac{1}{2}\right) - 2$$

$$= 3\left(-\frac{1}{8}\right) - 10\left(\frac{1}{4}\right) - \frac{9}{2} - 2$$

$$= -\frac{3}{8} - \frac{10}{4} - \frac{9}{2} - 2$$

$$= -\frac{75}{8}$$

2. Solve the equation $2\,cos^2x - 1 = \sin x \quad for\ 0 \le x \le 2\pi$. Give your answer in terms of π.

SOLUTION

$0 \le x \le 2\pi$

$2\,cos^2x - 1 = \sin x$

$2\,(1 - sin^2x) - 1 = \sin x$

$2 - 2\,sin^2x - 1 = \sin x$

$2\,sin^2x + \sin x\ - 1 = 0$

$\boldsymbol{Let\ u = sin\ x}$

$2\,u^2 + u - 1 = 0$

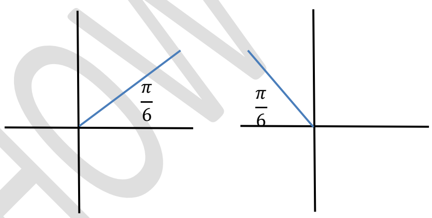
$(2u - 1)(u + 1) = 0$

$(2u - 1) = 0$

$u = \frac{1}{2}$

$\sin x = \frac{1}{2}$

$\alpha = sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$



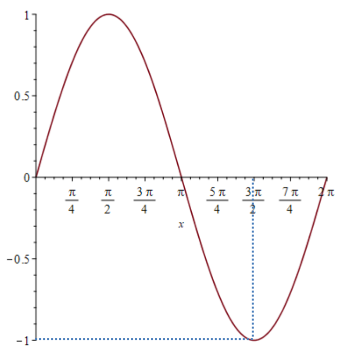
$x = \frac{\pi}{6}, \pi - \frac{\pi}{6}$

$x = \frac{\pi}{6}, \quad \frac{5\pi}{6}$

$(u + 1) = 0$

$u = -1$

$\sin x = -1$



$x = \frac{3\pi}{2}$

$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

3. Find the relative extremum of the curve $y = x^3 - 4x^2 + 4x$.

SOLUTION

$$y = x^3 - 4x^2 + 4x$$

$$\frac{dy}{dx} = 3x^2 - 8x + 4$$

$$Let\ \frac{dy}{dx} = 0$$

$$3x^2 - 8x + 4 = 0$$

$$(3x - 2)(x - 2) = 0$$

$$x = \frac{2}{3} \quad or \quad x = 2$$

$$\frac{d^2y}{dx^2} = 6x - 8$$

$$When\ x = \frac{2}{3}$$

$$y = \left(\frac{2}{3}\right)^3 - 4\left(\frac{2}{3}\right)^2 + 4\left(\frac{2}{3}\right)$$

$$= \frac{8}{27} - \frac{16}{9} + \frac{8}{3}$$

$$= \frac{32}{27}$$

$$\frac{d^2y}{dx^2} = 6\left(\frac{2}{3}\right) - 8$$

$$= -4 < 0\ (Max)$$

$$\therefore The\ point\ \left(\frac{2}{3}, \frac{32}{27}\right)\ is\ a\ relative\ maximum\ point.$$

$$When\ x = 2$$

$$y = (2)^3 - 4(2)^2 + 4(2)$$

$= 8 - 16 + 8$

$= 0$

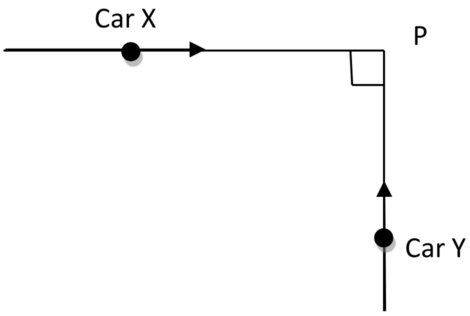
$\frac{d^2y}{dx^2} = 6(2) - 8$

$= 4 > 0\ (Min)$

$\therefore The\ point\ (2, 0)\ is\ a\ relative\ minimum\ point.$

4. Car X is travelling east at a speed of 80km/h and car Y is travelling north at 100km/h as shown in the diagram below. Obtain an equation that describes the rate of change of the distance between the two cars.

Hence, evaluate the rate of change of the distance between the two cars when car X is 0.15km and car Y is 0.08km from P.

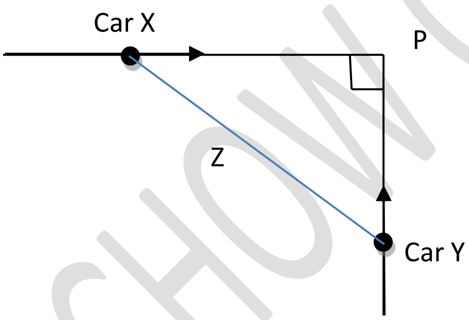


SOLUTION

$$\frac{dx}{dt} = -80$$

$$\frac{dy}{dt} = -100$$

Negative sign as the distance is decreasing.



$$z^2 = x^2 + y^2$$

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{1}{2z}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)$$

$$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$When\ x = 0.15, y = 0.08, \quad \frac{dx}{dt} = -80, \qquad \frac{dy}{dt} = -100$

$$z^2 = x^2 + y^2$$

$$z^2 = (0.15)^2 + (0.08)^2$$

$$z^2 = (0.15)^2 + (0.08)^2$$

$$z = 0.17$$

$$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$\frac{dz}{dt} = \frac{1}{0.17}[(0.15)(-80) + (0.08)(-100)]$$

$$= -117.647$$

The rate of change of the distance between the two cars is 117.65km/h

5. Expand $(x+a)(x+b)^2$, a and b are real numbers with $b>0$. Hence, find the values of a and b if $(x+a)(x+b)^2 = x^3-3x-2$.

Express $\frac{x^4-4x^2+5x-1}{x^3-3x-2}$ in the form of partial fractions.

SOLUTION

$(x+a)(x+b)^2 = (x+a)(x^2+2bx+b^2)$

$= x^3+2bx^2+b^2x+ax^2+2abx+ab^2$

$= x^3+(a+2b)x^2+(b^2+2ab)x+ab^2$

$If\ (x+a)(x+b)^2 = x^3-3x-2$

$x^3+(a+2b)x^2+(b^2+2ab)x+ab^2 = x^3-3x-2$

$a+2b=0$ .... (1)

$b^2+2ab=-3$ .... (2)

$ab^2=-2$ .... (3)

$From\ (1)$

$a=-2b$

$b^2+2(-2b)b=-3$

$b^2-4b^2=-3$

$-3b^2=-3$

$b^2=1$

$b=\pm 1$

$Since\ b>0, b=1$

$When\ b=1$

$a=-2(1)$

$=-2$

$a=-2, \quad b=1$

$$\frac{x^4-4x^2+5x-1}{x^3-3x-2}$$

$$\begin{array}{r} x \qquad\qquad\qquad\qquad \\ x^3-3x-2\overline{)x^4+0x^3-4x^2+5x-1} \\ \underline{x^4-0x^3-3x^2-2x\quad} \\ -x^2+7x-1 \end{array}$$

$$\frac{x^4-4x^2+5x-1}{x^3-3x-2}=x+\frac{-x^2+7x-1}{x^3-3x-2}$$

$$=x-\frac{x^2-7x+1}{(x-2)(x+1)^2}$$

$$\frac{x^2-7x+1}{(x-2)(x+1)^2}=\frac{A}{x-2}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$=\frac{A(x+1)^2+B(x-2)(x+1)+C(x-2)}{(x-2)(x+1)^2}$$

$$x^2-7x+1=A(x+1)^2+B(x-2)(x+1)+C(x-2)$$

$When\, x=-1$

$$x^2-7x+1=A(x+1)^2+B(x-2)(x+1)+C(x-2)$$

$$(-1)^2-7(-1)+1=C(-1-2)$$

$$9=-3C$$

$$C=-3$$

$When\, x=2$

$$x^2-7x+1=A(x+1)^2+B(x-2)(x+1)+C(x-2)$$

$$(2)^2-7(2)+1=A(2+1)^2$$

$$-9=9A$$

$$A=-1$$

$When\ x = 0, A = -1, C = -3$

$$x^2 - 7x + 1 = A(x+1)^2 + B(x-2)(x+1) + C(x-2)$$

$$1 = A - 2B - 2C$$

$$1 = -1 - 2B - 2(-3)$$

$$1 = 5 - 2B$$

$$2B = 4$$

$$B = 2$$

$$\frac{x^4 - 4x^2 + 5x - 1}{x^3 - 3x - 2} = x - \frac{x^2 - 7x + 1}{(x-2)(x+1)^2}$$

$$= x - \left[\frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}\right]$$

$$= x - \left[\frac{-1}{x-2} + \frac{2}{x+1} + \frac{-3}{(x+1)^2}\right]$$

$$= x + \frac{1}{x-2} - \frac{2}{x+1} + \frac{3}{(x+1)^2}$$

6. (a) Express $\sin 6x - \sin 2x$ in product form. Hence, show that

$\sin 6x - \sin 2x + \sin 4x = 4\cos 3x \sin 2x \cos x$.

(b) Use the result in (a) to solve

$\sin 6x - \sin 2x + \sin 4x = \sin 2x \cos x$ for $0 \leq x \leq 180°$.

SOLUTION

(a) Express $\sin 6x - \sin 2x$ in product form. Hence, show that

$\sin 6x - \sin 2x + \sin 4x = 4\cos 3x \sin 2x \cos x$.

$$\sin P - \sin Q = 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$$

$$\sin 6x - \sin 2x = 2\cos\left(\frac{6x+2x}{2}\right)\sin\left(\frac{6x-2x}{2}\right)$$

$$= 2\cos 4x \sin 2x$$

$$\sin 2x = 2\sin x \cos x$$

$$\sin 2(2x) = 2\sin(2x)\cos(2x)$$

$$\cos P + \cos Q = 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$$

$\sin 6x - \sin 2x + \sin 4x$

$= 2\cos 4x \sin 2x + \sin 4x$

$= 2\cos 4x \sin 2x + \sin 2(2x)$

$= 2\cos 4x \sin 2x + 2\sin 2x \cos 2x$

$= 2\sin 2x\,(\cos 4x + \cos 2x)$

$= 2\sin 2x\left[2\cos\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right)\right]$

$= 2\sin 2x\,[2\cos 3x \cos x]$

$= 4\cos 3x \sin 2x \cos x$

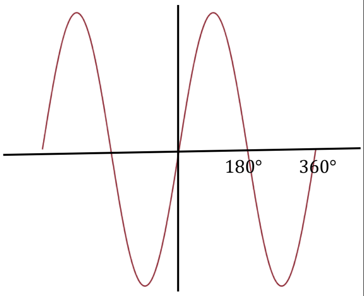
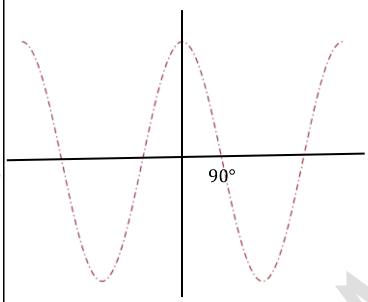
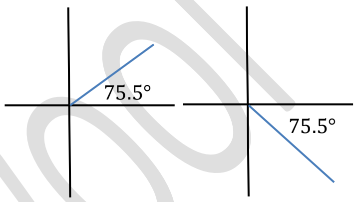
(b) Use the result in (a) to solve $\sin 6x - \sin 2x + \sin 4x = \sin 2x \cos x$.

$\sin 6x - \sin 2x + \sin 4x = \sin 2x \cos x$

$4\cos 3x \sin 2x \cos x = \sin 2x \cos x$

$4\cos 3x \sin 2x \cos x - \sin 2x \cos x = 0$

$\sin 2x \cos x\,(4\cos 3x - 1) = 0$

$0 \leq x \leq 180°$ $0 \leq 2x \leq 360°$ $\sin 2x = 0$  $2x = 0°, 180°, 360°$ $x = 0°, 90°, 180°$	$0 \leq x \leq 180°.$ $\cos x = 0$  $x = 90°$	$0 \leq x \leq 180°$ $0 \leq 3x \leq 540°$ $4\cos 3x - 1 = 0$ $\cos 3x = \frac{1}{4}$  $\alpha = 75.5°$ $3x = 75.5°, 284.5°, 435.5°$ $x = 25.2°, 94.8°, 145.2°$
$\therefore x = 0°, 25.2°, 90°, 94.8°, 145.2°, 180°$		

7. Find the limit of the following, if it exists.

a. $\lim_{x \to -3} \frac{x+3}{x^3+27}$

b. $\lim_{x \to -\infty} \frac{2x-1}{\sqrt{x^2-9}}$

c. $\lim_{x \to 4} \frac{x^2-3x-4}{\sqrt{x}-2}$

SOLUTION

a. $\lim_{x \to -3} \frac{x+3}{x^3+27} = \lim_{x \to -3} \frac{x+3}{(x+3)(x^2-3x+9)}$

$$= \lim_{x \to -3} \frac{1}{(x^2-3x+9)}$$

$$= \frac{1}{(-3)^2 - 3(-3) + 9}$$

$$= \frac{1}{27}$$

$$\begin{array}{r} x^2 - 3x + 9 \\ x+3 \overline{\big)\, x^3 + 0x^2 + 0x + 27} \\ x^3 + 3x^2 + 0x \\ -3x^2 + 0x + 27 \\ \underline{-3x^2 - 9x} \\ 9x + 27 \\ \underline{9x + 27} \end{array}$$

b. $\lim_{x \to -\infty} \frac{2x-1}{\sqrt{x^2-9}} = \lim_{x \to -\infty} \frac{\frac{2x}{-x} - \frac{1}{-x}}{\sqrt{\frac{x^2}{x^2} - \frac{9}{x^2}}}$

$$= \lim_{x \to -\infty} \frac{-2 + \frac{1}{x}}{\sqrt{1 - \frac{9}{x^2}}}$$

$$= \frac{-2}{\sqrt{1}}$$

$$= -2$$

c. $\lim_{x \to 4} \frac{x^2-3x-4}{\sqrt{x}-2} = \lim_{x \to 4} \frac{(x-4)(x+1)}{\sqrt{x}-2}$

$$= \lim_{x \to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)(x+1)}{\sqrt{x}-2}$$

$$= \lim_{x \to 4} (\sqrt{x}+2)(x+1)$$

$$= (\sqrt{4}+2)(4+1)$$

$$= 20$$

$$a^2 - b^2 = (a-b)(a+b)$$

$$x - 4 = \sqrt{x}^2 - 2^2$$

$$= (\sqrt{x}-2)(\sqrt{x}+2)$$

8. Given that $f(x) = \begin{cases} 1+e^x & x \leq 0 \\ \frac{x+6}{3-x} & 0 < x \leq 4 \\ C & x > 4 \end{cases}$

Where C is a constant.

a) Determine whether $f(x)$ is continuous at $x = 0$.

b) Given that $f(x)$ is discontinuous at $x = 4$, determine the values of C.

c) Find the vertical asymptote of $f(x)$.

SOLUTION

$$f(x) = \begin{cases} 1+e^x & x \leq 0 \\ \dfrac{x+6}{3-x} & 0 < x \leq 4 \\ C & x > 4 \end{cases}$$

a) Determine whether $f(x)$ is continuous at $x = 0$

$$f(0) = 1 + e^0 = 2$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 1 + e^x$$

$$= 1 + e^0$$

$$= 2$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x+6}{3-x}$$

$$= \frac{0+6}{3-0}$$

$$= 2$$

$$\lim_{x \to 0} f(x) = 2$$

$$Since\ f(0) = \lim_{x \to 0} f(x) = 2,\ therefore\ f(x) is\ continuous\ at\ x = 0.$$

Continuity at a point

$If\ f(x) is\ continuous\ at\ x = c\ then$

1. $f(c) defined$
2. $\lim_{x \to c} f(x)\ exist$
3. $\lim_{x \to c} f(x) = f(c)$

b) Given that $f(x)$ is discontinuous at $x = 4$, determine the values of C.

$$At\ x = 4$$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \frac{x+6}{3-x}$$

$$= \frac{4+6}{3-4}$$

$$= \frac{10}{-1}$$

$$= -10$$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} C$$

$$= C$$

$$Since\ f(x) is\ discontinuous\ at\ x = 4, therefore\ C \neq -10.$$

$$\{C: C < -10\ \cup C > -10\}$$

c) Find the vertical asymptote of $f(x)$.

$\boldsymbol{Vertical\ Asymptote}$:

$$3 - x = 0$$

$$x = 3$$

$$\lim_{x\to 3^+} \frac{x+6}{3-x} = -\infty$$

$$\lim_{x\to 3^-} \frac{x+6}{3-x} = \infty$$

$$\therefore the\ vertical\ asymtote\ is\ x = 3.$$

9. Consider the parametric equations of the curve

$x = cos^3\theta$ and $y = sin^3\theta,$ $0 < \theta \leq 2\pi.$

a) Find $\frac{dy}{dx}$ and express your answer in terms of θ.

b) Find the value of $\frac{dy}{dx}$ if $x = \frac{\sqrt{2}}{4}$

c) Show that $\frac{d^2y}{dx^2} = \frac{1}{3cos^4\theta \sin\theta}$.

Hence, calculate $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{3}$.

SOLUTION

a) Find $\frac{dy}{dx}$ and express your answer in terms of θ.

$x = cos^3\theta$

$$\frac{dx}{d\theta} = 3\cos^2\theta \frac{d}{d\theta}(\cos\theta)$$

$$= 3\cos^2\theta(-\sin\theta)$$

$$= -3\sin\theta\cos^2\theta$$

$y = sin^3\theta$

$$\frac{dy}{d\theta} = 3\sin^2\theta \frac{d}{d\theta}(\sin\theta)$$

$$= 3\sin^2\theta(\cos\theta)$$

$$= 3\cos\theta\sin^2\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx}$$

$$= (3\cos\theta\sin^2\theta)\left(\frac{1}{-3\sin\theta\cos^2\theta}\right)$$

$$= (\sin\theta)\left(\frac{1}{-\cos\theta}\right)$$

$$= -\frac{\sin\theta}{\cos\theta}$$

$$= -\tan\theta$$

b) Find the value of $\frac{dy}{dx}$ if $x = \frac{\sqrt{2}}{4}$

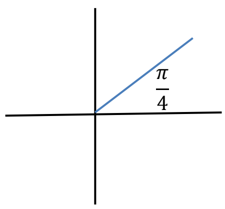
$$When\ x = \frac{\sqrt{2}}{4}$$

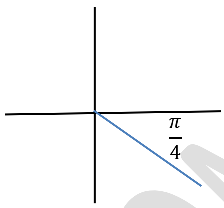
$$x = cos^3\theta$$

$$cos^3\theta = \frac{\sqrt{2}}{4}$$

$$\cos\theta = \left(\frac{\sqrt{2}}{4}\right)^{\frac{1}{3}}$$

$$\approx 0.7071$$





$$Basic\ Angle, \alpha = \frac{\pi}{4}$$

$$\theta = \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$$

$$\theta = \frac{\pi}{4}, \frac{7\pi}{4}$$

$$When\ x = \frac{\sqrt{2}}{4}, \theta = \frac{\pi}{4}$$

$$\frac{dy}{dx} = -\tan\theta$$

$$= -\tan\left(\frac{\pi}{4}\right)$$

$$= -1$$

$$When\ x = \frac{\sqrt{2}}{4}, \theta = \frac{7\pi}{4}$$

$$\frac{dy}{dx} = -\tan\theta$$

$$= -\tan\left(\frac{7\pi}{4}\right)$$

$$= -(-1)$$

$$= 1$$

c) Show that $\frac{d^2y}{dx^2} = \frac{1}{3cos^4\theta \sin\theta}$. Hence, calculate $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{3}$.

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left[\frac{dy}{dx}\right] . \frac{d\theta}{dx}$$

$$= \left[\frac{d}{d\theta}(-\tan\theta)\right] . \left(\frac{1}{-3\sin\theta\cos^2\theta}\right)$$

$$= (-sec^2\,\theta) . \left(\frac{1}{-3\sin\theta\cos^2\theta}\right)$$

$$= \frac{sec^2\,\theta}{3\sin\theta\cos^2\theta}$$

$$= \frac{\frac{1}{\cos^2\theta}}{3\sin\theta\cos^2\theta}$$

$$= \frac{1}{3\cos^4\theta\sin\theta}$$

$$At\ \theta = \frac{\pi}{3}$$

$$\frac{d^2y}{dx^2} = \frac{1}{3\cos^4\theta\sin\theta}$$

$$= \frac{1}{3\cos^4\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)}$$

$$= 6.16$$

10. (a) Use the first principle to find the derivative of $g(x) = \sqrt{1-x}$.

(b) Given that $e^y + xy + \ln(1+2x) = 1, x \geq 0$

Show that $(e^y + x)\frac{d^2y}{dx^2} + e^y\left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx} - \frac{4}{(1+2x)^2} = 0$.

Hence, find the value of $\frac{d^2y}{dx^2}$ at the point $(0,0)$.

SOLUTION

(a) $g(x) = \sqrt{1-x}$

$$g(x+h) = \sqrt{1-(x+h)}$$

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{1-(x+h)} - \sqrt{1-x}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{1-(x+h)} - \sqrt{1-x}}{h}\cdot\frac{\sqrt{1-(x+h)} + \sqrt{1-x}}{\sqrt{1-(x+h)} + \sqrt{1-x}}$$

$$= \lim_{h\to 0}\frac{[1-(x+h)]-(1-x)}{h\left(\sqrt{1-(x+h)} + \sqrt{1-x}\right)}$$

$$= \lim_{h\to 0}\frac{1-x-h-1+x}{h\left(\sqrt{1-(x+h)} + \sqrt{1-x}\right)}$$

$$= \lim_{h\to 0}\frac{-h}{h\left(\sqrt{1-(x+h)} + \sqrt{1-x}\right)}$$

$$= \lim_{h\to 0}\frac{-1}{\left(\sqrt{1-(x+h)} + \sqrt{1-x}\right)}$$

$$= \frac{-1}{\left(\sqrt{1-(x+0)} + \sqrt{1-x}\right)}$$

$$= \frac{-1}{\left(\sqrt{1-x} + \sqrt{1-x}\right)}$$

$$= \frac{-1}{2\sqrt{1-x}}$$

(b) $e^y + xy + \ln(1+2x) = 1$

$$e^y \frac{dy}{dx} + \left[x\frac{dy}{dx} + y\right] + \frac{1}{(1+2x)}\frac{d}{dx}(1+2x) = 0$$

$$e^y \frac{dy}{dx} + x\frac{dy}{dx} + y + \frac{1}{(1+2x)}(2) = 0$$

$$e^y \frac{dy}{dx} + x\frac{dy}{dx} + y + \frac{2}{(1+2x)} = 0$$

$$(e^y + x)\frac{dy}{dx} + y + \frac{2}{(1+2x)} = 0$$

$$(e^y + x)\frac{dy}{dx} + y + 2(1+2x)^{-1} = 0$$

$$\left[(e^y + x)\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(e^y\frac{dy}{dx} + 1\right)\right] + \frac{dy}{dx} - 2(1+2x)^{-2}\frac{d}{dx}(1+2x) = 0$$

$$\left[(e^y + x)\frac{d^2y}{dx^2} + e^y\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx}\right] + \frac{dy}{dx} - \frac{2}{(1+2x)^2}(2) = 0$$

$$(e^y + x)\frac{d^2y}{dx^2} + e^y\left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx} - \frac{4}{(1+2x)^2} = 0$$

At the point $(0,0)$ ➔ $x = 0,\ \ y = 0$

$$(e^0 + 0)\frac{dy}{dx} + 0 + \frac{2}{(1+2(0))} = 0$$

$$(1+0)\frac{dy}{dx} + \frac{2}{(1)} = 0$$

$$\frac{dy}{dx} = -2$$

When $x = 0, \quad y = 0, \quad \frac{dy}{dx} = -2$

$$(e^0 + 0)\frac{d^2y}{dx^2} + e^0(-2)^2 + 2(-2) - \frac{4}{(1+2(0))^2} = 0$$

$$(1+0)\frac{d^2y}{dx^2} + 4 - 4 - \frac{4}{1} = 0$$

$$\frac{d^2y}{dx^2} = 4$$