KOLEJ MATRIKULASI KEDAH
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QS 025/1

Matriculation Programme Examination

Semester II

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1. Evaluate $\int_0^{\frac{\pi}{6}} \tan 2\theta \; cos^2 2\theta \; d\theta$.

2. Given vectors $\boldsymbol{p} = 3\boldsymbol{i} - 6\boldsymbol{j} + \alpha\boldsymbol{k}$ and $\boldsymbol{q} = \beta\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k}$ where α and β are contants.
   a) Find the values of α and β if $\boldsymbol{p}$ and $\boldsymbol{q}$ are parellel.
   b) Given $\alpha = 1$, find β if $\boldsymbol{p}$ and $\boldsymbol{q}$ are perpendicular.

3. Given three points P(-3, 2, -1), Q(-2, 4, 5) and R(1, -2, 4). Calculate the area of triangle PQR.

4. Determine the vertices and foci of the ellipse $25x^2 + 4y^2 - 250x - 16y + 541 = 0$. Sketch the ellipse and lable the focy, center and vertices.

5. Show that the equation $\ln x + x - 4 = 0$ has a root between 1 and 3. From the Newton-Raphson formula, show that iterative equation of the root is $x_{n+1} = \frac{x_n(5-\ln x_n)}{1+x_n}$. Hence, if the initial value is $x_1 = 2$, calculate the root correct to three decimal places.

6. Show that the line $2y - 5x + 4 = 0$ does not intersect the circle $x^2 + y^2 + 3x - 2y + 2 = 0$. Find centre and radius of the circle. Hence, determine the shortest distance between the line and the circle.

7. Given the line L: $\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-2}{-3}$ and the planes $\pi_1: 2x - y - 2z = 17$ and $\pi_2: -4x - 3y + 5z = 10$.

   Find

   a) The intersection point between L and π_1.
   b) The acute angle between π_1 and π_2.
   c) The parametric equations of the line that passes through the point $(2, -1, 3)$ and perpendicular to the plance π_2.

8. Express $\frac{6x^2-x+7}{(4-3x)(1+x)^2}$ in partial fractions. Hence, show that $\int_0^1 \frac{6x^2-x+7}{(4-3x)(1+x)^2} = 1 + \ln 2$.

9. a) Find the general solution of the differential equation $\frac{dy}{dx} = y^2xe^{-2x}$. Give your answer in the form $y = f(x)$.

   b) Find the particular solution of the differential equation $\frac{dy}{dx} + \frac{xy}{1+x^2} = \sqrt{1+x^2}$, given that $y = 1$ when $x = 0$.

10. Given the curve $y^2 = x$ and the line $y = -2x + 1$.
    a) Determine the points of intersection between the curve and the line.

b) Sketch the curve and the line on the same axes. Shade the region R bounded by the curve and the line. Label the points of intersection.

c) Find the area of the region R.

d) Calculate the volume of the solid generated when the region R is rotated 2π radian about the y-axis.

END OF QUESTION PAPER

1. Evaluate $\int_0^{\frac{\pi}{6}} \tan 2\theta \ cos^2 2\theta \, d\theta$.

SOLUTION

$$\int_0^{\frac{\pi}{6}} \tan 2\theta \ cos^2 2\theta \ d\theta = \int_0^{\frac{\pi}{6}} \frac{\sin 2\theta}{\cos 2\theta} \ cos^2 2\theta \ d\theta$$

$$= \int_0^{\frac{\pi}{6}} \sin 2\theta \cos 2\theta \ \ d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{2} \sin 2(2\theta) \ \ d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} \sin 4\theta \ \ d\theta$$

$$= -\frac{1}{8}[\cos 4\theta]_0^{\frac{\pi}{6}}$$

$$= -\frac{1}{8}\left\{\left[\cos 4\left(\frac{\pi}{6}\right)\right] - [\cos 4(0)]\right\}$$

$$= -\frac{1}{8}\left[\cos \frac{2\pi}{3} - 1\right]$$

$$= -\frac{1}{8}\left[-\frac{1}{2} - 1\right]$$

$$= -\frac{1}{8}\left[-\frac{3}{2}\right]$$

$$= \frac{3}{16}$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

$$\sin 2\theta \cos 2\theta = \frac{1}{2}\sin 2(2\theta)$$

2. Given vectors $\boldsymbol{p} = 3\boldsymbol{i} - 6\boldsymbol{j} + \alpha\boldsymbol{k}$ and $\boldsymbol{q} = \beta\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k}$ where α and β are contants.
   a) Find the values of α and β if $\boldsymbol{p}$ and $\boldsymbol{q}$ are parellel.
   b) Given $\alpha = 1$, find β if $\boldsymbol{p}$ and $\boldsymbol{q}$ are perpendicular.

SOLUTION

$$\boldsymbol{p} = 3\boldsymbol{i} - 6\boldsymbol{j} + \alpha\boldsymbol{k}$$

$$\boldsymbol{q} = \beta\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k}$$

a) $\boldsymbol{p}$ and $\boldsymbol{q}$ are parellel ➔ $\boldsymbol{p}\ \boldsymbol{x}\ \boldsymbol{q} = \boldsymbol{0}.$

$$\boldsymbol{p}\ \boldsymbol{x}\ \boldsymbol{q} = \boldsymbol{0}$$

$$\begin{vmatrix} i & j & k \\ 3 & -6 & \alpha \\ \beta & -4 & 5 \end{vmatrix} = 0$$

$$(-30 + 4\alpha)\boldsymbol{i} - (15 - \alpha\beta)\boldsymbol{j} + (-12 + 6\beta)\boldsymbol{k} = 0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}$$

$-30 + 4\alpha = 0$ ➔ $\alpha = \frac{30}{4} = \frac{15}{2}$

$-12 + 6\beta = 0$ ➔ $\beta = \frac{12}{6} = 2$

b) Given $\alpha = 1$. $\boldsymbol{p}$ and $\boldsymbol{q}$ are perpendicular➔ $\boldsymbol{p}\,.\,\boldsymbol{q} = \boldsymbol{0}$

$$\boldsymbol{p}\,.\,\boldsymbol{q} = \boldsymbol{0}$$

$$(3\boldsymbol{i} - 6\boldsymbol{j} + \boldsymbol{k})\,.\,(\beta\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k}) = \boldsymbol{0}$$

$$(3)(\beta) + (-6)(-4) + (1)(5) = 0$$

$$3\beta + 24 + 5 = 0$$

$$3\beta = -29$$

$$\beta = \frac{-29}{3}$$

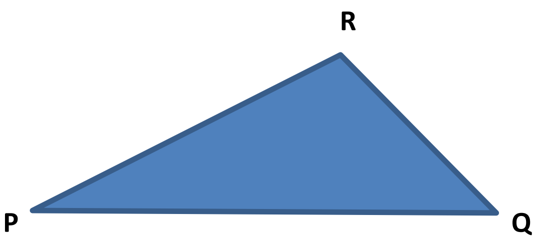
3. Given three points P(-3, 2, -1), Q(-2, 4, 5) and R(1, -2, 4). Calculate the area of triangle PQR.

SOLUTION

P(-3, 2, -1) ➔ $\overrightarrow{OP} = -3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

Q(-2, 4, 5) ➔ $\overrightarrow{OQ} = -2\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k}$

R(1, -2, 4) ➔ $\overrightarrow{OR} = \boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}$



$Area = \frac{1}{2}\left|\overrightarrow{PQ}\ x\ \overrightarrow{PR}\right|$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= (-2\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k}) - (-3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$

$= \boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP}$

$= (\ \boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}) - (-3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$

$= 4\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k}$

$$\overrightarrow{PQ}\ x\ \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & 6 \\ 4 & -4 & 5 \end{vmatrix}$$

$= (10 + 24)\boldsymbol{i} - (5 - 24)\boldsymbol{j} + (-4 - 8)\boldsymbol{k}$

$$= 34\boldsymbol{i} + 19\boldsymbol{j} - 12\boldsymbol{k}$$

$$\left|\overrightarrow{PQ}\ x\ \overrightarrow{PR}\right| = \sqrt{(34)^2 + (19)^2 + (-12)^2}$$

$$= \sqrt{1661}$$

$$Area = \frac{1}{2}\left|\overrightarrow{PQ}\ x\ \overrightarrow{PR}\right|$$

$$= \frac{1}{2}\sqrt{1661}$$

$$= 20.38\ unit^2$$

4. Determine the vertices and foci of the ellipse $25x^2 + 4y^2 - 250x - 16y + 541 = 0$. Sketch the ellipse and lable the focy, center and vertices.

SOLUTION

$$25x^2 + 4y^2 - 250x - 16y + 541 = 0$$

$$25x^2 - 250x + 4y^2 - 16y = -541$$

$$25(x^2 - 10x) + 4(y^2 - 4y) = -541$$

$$25\left[x^2 - 10x + \left(\frac{-10}{2}\right)^2 - \left(\frac{-10}{2}\right)^2\right] + 4\left[y^2 - 4y + \left(\frac{-4}{2}\right)^2 - \left(\frac{-4}{2}\right)^2\right] = -541$$

$$25[(x-5)^2 - (-5)^2] + 4[(y-2)^2 - (-2)^2] = -541$$

$$25[(x-5)^2 - 25] + 4[(y-2)^2 - 4] = -541$$

$$25(x-5)^2 - 625 + 4(y-2)^2 - 16 = -541$$

$$25(x-5)^2 + 4(y-2)^2 = -541 + 625 + 16$$

$$25(x-5)^2 + 4(y-2)^2 = 100$$

$$\frac{25(x-5)^2}{100} + \frac{4(y-2)^2}{100} = \frac{100}{100}$$

$$\frac{(x-5)^2}{4} + \frac{(y-2)^2}{25} = 1$$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$h = 5, k = 2, a = 2, b = 5$$

$$c^2 = b^2 - a^2$$

$$= 5^2 - 2^2$$

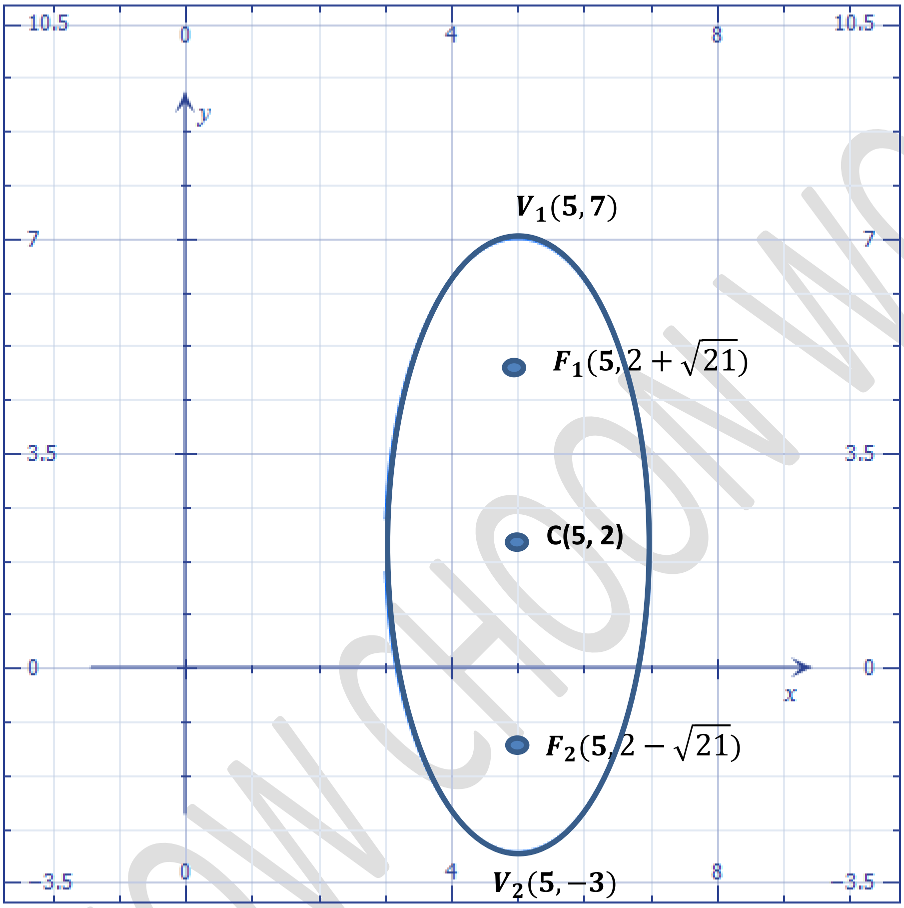
$$= 21$$

$$c = \sqrt{21}$$

$Center\ (h, k) = (5, 2)$

$Vertices, (h, k \pm b) = (5, 2 \pm 5) = (5,7)\ and\ (5, -3)$

$Foci, (h, k \pm c) = \left(5, 2 \pm \sqrt{21}\right) = \left(5, 2 + \sqrt{21}\right)\ and\ \left(5, 2 - \sqrt{21}\right)$



5. Show that the equation $\ln x + x - 4 = 0$ has a root between 1 and 3. From the Newton-Raphson formula, show that iterative equation of the root is $x_{n+1} = \frac{x_n(5-\ln x_n)}{1+x_n}$. Hence, if the initial value is $x_1 = 2$, calculate the root correct to three decimal places.

SOLUTION

$\ln x + x - 4 = 0$

Let

$f(x) = \ln x + x - 4$

$f(1) = \ln(1) + (1) - 4 = -3 < 0$

$f(3) = \ln(3) + (3) - 4 = 0.099 > 0$

$Since\ f(1) < 0\ and\ f(3) > 0, therefore\ \ln x + x - 4 = 0\ \ has\ root\ between\ 1\ and\ 3.$

$$f(x) = \ln x + x - 4$$

$$f'(x) = \frac{1}{x} + 1$$

$$= \frac{1+x}{x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\ln x_n + x_n - 4}{\frac{1+x_n}{x_n}}$$

$$= x_n - \frac{x_n(\ln x_n + x_n - 4)}{1+x_n}$$

$$= \frac{x_n(1+x_n) - x_n(\ln x_n + x_n - 4)}{1+x_n}$$

$$= \frac{x_n[1 + x_n - (\ln x_n + x_n - 4\,)]}{1 + x_n}$$

$$= \frac{x_n[1 + x_n - \ln x_n - x_n + 4\,]}{1 + x_n}$$

$$= \frac{x_n[5 - \ln x_n]}{1 + x_n}$$

$x_1 = 2$

$$x_2 = \frac{2[5 - \ln 2]}{1 + 2} = 2.8712$$

$$x_3 = \frac{2.8712[5 - \ln 2.8712]}{1 + 2.8712} = 2.9261$$

$$x_4 = \frac{2.9261[5 - \ln 2.9261]}{1 + 2.9261} = 2.9263$$

$\therefore x = 2.926$

6. Show that the line $2y - 5x + 4 = 0$ does not intersect the circle $x^2 + y^2 + 3x - 2y + 2 = 0$. Find centre and radius of the circle. Hence, determine the shortest distance between the line and the circle.

SOLUTION

$2y - 5x + 4 = 0$

$y = \frac{5x-4}{2}$ .... (1)

$x^2 + y^2 + 3x - 2y + 2 = 0$ .... (2)

Substitute (1) into (2)

$$x^2 + \left(\frac{5x-4}{2}\right)^2 + 3x - 2\left(\frac{5x-4}{2}\right) + 2 = 0$$

$$x^2 + \frac{25x^2 - 40x + 16}{4} + 3x - (5x - 4) + 2 = 0$$

$$4x^2 + 25x^2 - 40x + 16 + 12x - 20x + 16 + 8 = 0$$

$$29x^2 - 48x + 40 = 0$$

$$b^2 - 4ac = 48^2 - 4(29)(40)$$

$$= -2336 < 0$$

$Since\ b^2 - 4ac < 0,$

$therefore$ $2y - 5x + 4 = 0$ does not intersect the circle $x^2 + y^2 + 3x - 2y + 2 = 0$

$x^2 + y^2 + 3x - 2y + 2 = 0$

$x^2 + y^2 + 2gx + 2fy + c = 0$

$2g = 3$ $2f = -2$ $c = 2$

$g = \frac{3}{2}$ $f = -1$

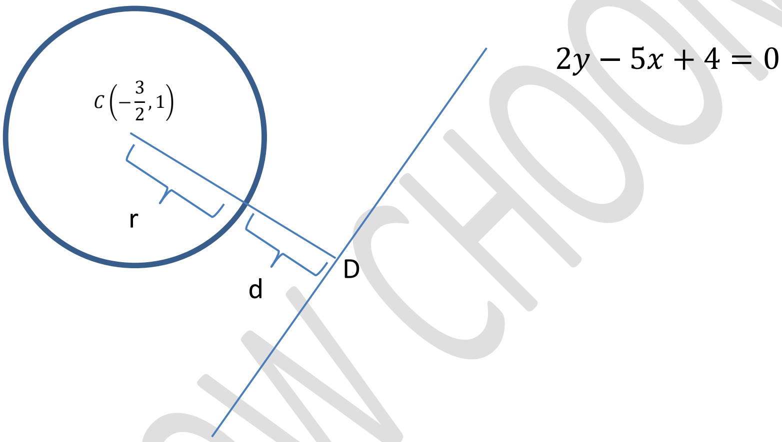
$$Centre\ of\ circle: (-g,-f) = \left(-\frac{3}{2}, 1\right)$$

$$radius: r = \sqrt{g^2 + f^2 - c}$$

$$= \sqrt{\left(\frac{3}{2}\right)^2 + (-1)^2 - 2}$$

$$= \sqrt{\frac{5}{4}}$$

$$= \frac{\sqrt{5}}{2}$$



$d = CD - r$

$$CD = \frac{|ah + bk + c|}{\sqrt{(a)^2 + (b)^2}}$$

$$= \frac{\left|(-5)\left(-\frac{3}{2}\right) + (2)(1) + (4)\right|}{\sqrt{(2)^2 + (-5)^2}}$$

$$= \frac{\left|\frac{27}{2}\right|}{\sqrt{4 + 25}}$$

$$= \frac{27}{2\sqrt{29}}$$

$$d = CD - r$$

$$= \frac{27}{2\sqrt{29}} - \frac{\sqrt{5}}{2}$$

$$= 1.39$$

7. Given the line $L: \frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-2}{-3}$ and the planes $\pi_1: 2x - y - 2z = 17$ and $\pi_2: -4x - 3y + 5z = 10$.

Find

a) The intersection point between L and π_1.
b) The acute angle between π_1 and π_2.
c) The parametric equations of the line that passes through the point $(2, -1, 3)$ and perpendicular to the plance π_2.

SOLUTION

a) $L: \frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-2}{-3}$ ………… (1)

$\pi_1: 2x - y - 2z = 17$ ………… (2)

From (1), the parametric equation of L:

$x = 1 + 2t;$ $y = 3 - t$ $z = 2 - 3t$ ………… (3)

Substitute (3) *into* (2)

$2(1 + 2t) - (3 - t) - 2(2 - 3t) = 17$

$2 + 4t - 3 + t - 4 + 6t = 17$

$11t = 22$

$t = 2$ ………… (4)

Substitute (4) *into* (3)

$x = 1 + 2(2);$ $y = 3 - 2$ $z = 2 - 3(2)$

$x = 5$ $y = 1$ $z = -4$

$The\ intersection\ point\ between\ L\ and\ \pi_1\ is\ (5,1,-4)$

b) $\pi_1: 2x - y - 2z = 17 \quad \rightarrow \quad \boldsymbol{n_1} = 2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$

$\pi_2: -4x - 3y + 5z = 10 \quad \rightarrow \quad \boldsymbol{n_2} = -4\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k}$

$$\cos\theta = \frac{\boldsymbol{n_1}.\boldsymbol{n_2}}{|\boldsymbol{n_1}||\boldsymbol{n_2}|}$$

$\boldsymbol{n_1}.\boldsymbol{n_2} = (2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}).(-4\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k})$

$= \mathbf{(2)(-4) + (-1)(-3) + (-2)(5)}$

$= \mathbf{-8 + 3 - 10}$

$= \mathbf{-15}$

$|\boldsymbol{n_1}| = \sqrt{(2)^2 + (-1)^2 + (-2)^2}$

$= 3$

$|\boldsymbol{n_1}| = \sqrt{(-4)^2 + (-3)^2 + (5)^2}$

$= \sqrt{50}$

$$\cos\theta = \frac{\boldsymbol{n_1}.\boldsymbol{n_2}}{|\boldsymbol{n_1}||\boldsymbol{n_2}|}$$

$$= \frac{-15}{3\sqrt{50}}$$

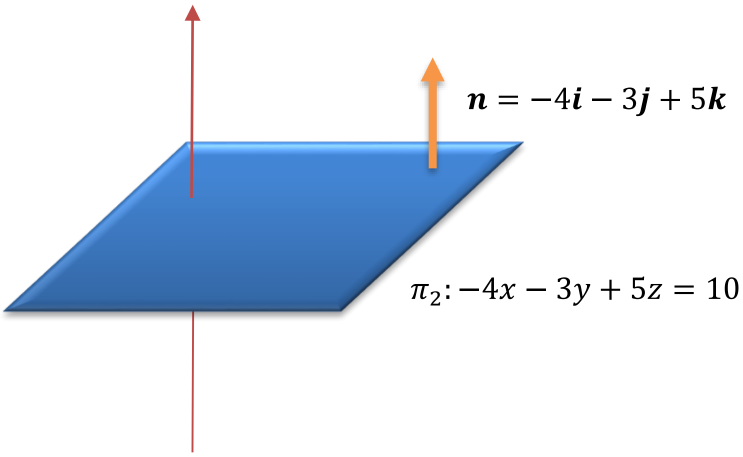
$= -0.7071$

$\theta = cos^{-1}(-0.7071)$

$= 135°$

$\therefore \theta = 180° - 135° = 45°$

c)



***Parametric equations of line*:**

$$x = x_1 + ta \qquad y = y_1 + tb \qquad z = z_1 + tc$$

$\boldsymbol{v} = \boldsymbol{n} = -4\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k}$ ➔ a=-4; b= -3; c=5

$\boldsymbol{a} = 2\boldsymbol{i} - 1\boldsymbol{j} + 3\boldsymbol{k}$ ➔ $x_1 = 2$; $y_1 = -1$; $z_1 = 3$

$$\therefore x = 2 - 4t \qquad y = -1 - 3t \qquad z = 3 + 5t$$

8. Express $\frac{6x^2-x+7}{(4-3x)(1+x)^2}$ in partial fractions. Hence, show that $\int_0^1 \frac{6x^2-x+7}{(4-3x)(1+x)^2}\,dx = 1+\ln 2$.

SOLUTION

$$\frac{6x^2-x+7}{(4-3x)(1+x)^2} = \frac{A}{(4-3x)} + \frac{B}{(1+x)} + \frac{C}{(1+x)^2}$$

$$\frac{6x^2-x+7}{(4-3x)(1+x)^2} = \frac{A(1+x)^2 + B(4-3x)(1+x) + C(4-3x)}{(4-3x)(1+x)^2}$$

$$6x^2-x+7 = A(1+x)^2 + B(4-3x)(1+x) + C(4-3x)$$

$When\ x = -1$

$$6x^2-x+7 = A(1+x)^2 + B(4-3x)(1+x) + C(4-3x)$$

$$6(-1)^2-(-1)+7 = A\left(1+(-1)\right)^2 + B\left(4-3(-1)\right)\left(1+(-1)\right) + C\left(4-3(-1)\right)$$

$$6+1+7 = A(0)^2 + B\left(4-3(-1)\right)(0) + C(4+3)$$

$$14 = 7C$$

$$C = 2$$

$When\ x = \frac{4}{3}$

$$6x^2-x+7 = A(1+x)^2 + B(4-3x)(1+x) + C(4-3x)$$

$$6\left(\frac{4}{3}\right)^2 - \left(\frac{4}{3}\right) + 7 = A\left[1+\left(\frac{4}{3}\right)\right]^2 + B\left[4-3\left(\frac{4}{3}\right)\right]\left[1+\left(\frac{4}{3}\right)\right] + C\left[4-3\left(\frac{4}{3}\right)\right]$$

$$6\left(\frac{16}{9}\right) - \left(\frac{4}{3}\right) + 7 = A\left[\frac{7}{3}\right]^2 + B[0]\left[1+\left(\frac{4}{3}\right)\right] + C[0]$$

$$\frac{32}{3} - \left(\frac{4}{3}\right) + \frac{21}{3} = \frac{49}{9}A$$

$$\frac{49}{3} = \frac{49}{9}A$$

$$A = \left(\frac{49}{3}\right)\left(\frac{9}{49}\right)$$

$$= 3$$

$Let\ A = 3, C = 2, x = 0$

$$6x^2 - x + 7 = A(1+x)^2 + B(4-3x)(1+x) + C(4-3x)$$

$$7 = 3(1)^2 + B(4)(1) + 2(4)$$

$$7 = 3 + 4B + 8$$

$$4B = -4$$

$$B = -1$$

$$\frac{6x^2 - x + 7}{(4-3x)(1+x)^2} = \frac{3}{(4-3x)} - \frac{1}{(1+x)} + \frac{2}{(1+x)^2}$$

$$\int \frac{6x^2 - x + 7}{(4-3x)(1+x)^2}\,dx = \int \frac{3}{(4-3x)} - \frac{1}{(1+x)} + \frac{2}{(1+x)^2}\,dx$$

$$= \int \frac{3}{(4-3x)}\,dx - \int \frac{1}{(1+x)}\,dx + \int \frac{2}{(1+x)^2}\,dx$$

$$= -\int \frac{-3}{(4-3x)}\,dx - \int \frac{1}{(1+x)}\,dx + \int 2(1+x)^{-2}\,dx$$

$$= -\ln(4-3x) - \ln(1+x) - \frac{2}{1+x}$$

$$\int_0^1 \frac{6x^2 - x + 7}{(4-3x)(1+x)^2}\,dx = \left[-\ln(4-3x) - \ln(1+x) - \frac{2}{1+x}\right]_0^1$$

$$= \left[-\ln(4-3(1)) - \ln(1+(1)) - \frac{2}{1+1}\right] - \left[-\ln(4-3(0)) - \ln(1+(0)) - \frac{2}{1+(0)}\right]$$

$$= \left[-\ln(1) - \ln(2) - \frac{2}{2}\right] - \left[-\ln(4) - \ln(1) - \frac{2}{1}\right]$$

$$= [0 - \ln 2 - 1] - [-\ln(4) - 0 - 2]$$

$$= 0 - \ln 2 - 1 + \ln 4 + 2$$

$$= -\ln 2 - 1 + \ln 2^2 + 2$$

$$= -\ln 2 + 2\ln 2 + 1$$

$$= 1 + \ln 2$$

9. a) Find the general solution of the differential equation $\frac{dy}{dx} = y^2xe^{-2x}$. Give your answer in the form $y = f(x)$.

b) Find the particular solution of the differential equation $\frac{dy}{dx} + \frac{xy}{1+x^2} = \sqrt{1+x^2}$, given that y = 1 when x = 0.

solution

a) $\frac{dy}{dx} = y^2xe^{-2x}$

$$\frac{dy}{y^2} = xe^{-2x}\,dx$$

$$\int \frac{1}{y^2}dy = \int xe^{-2x}\,dx$$

$$\int y^{-2}\,dy = \int xe^{-2x}\,dx$$

Integration by part

$u = x$ $\frac{du}{dx}=1$ $du = dx$	$dv = \int e^{-2x}\,dx$ $v = \frac{e^{-2x}}{-2}$

$$-\frac{1}{y} = uv - \int vdu$$

$$-\frac{1}{y} = (x)\left(\frac{e^{-2x}}{-2}\right) - \int \frac{e^{-2x}}{-2}dx$$

$$-\frac{1}{y} = \frac{xe^{-2x}}{-2} + \frac{1}{2}\int e^{-2x}dx$$

$$-\frac{1}{y} = \frac{xe^{-2x}}{-2} + \frac{1}{2}\left[\frac{e^{-2x}}{-2}\right] + c$$

$$-\frac{1}{y} = \frac{xe^{-2x}}{-2} - \frac{e^{-2x}}{4} + c$$

$$\frac{1}{y} = \frac{xe^{-2x}}{2} + \frac{e^{-2x}}{4} - c$$

$$\frac{1}{y} = \frac{2xe^{-2x}}{4} + \frac{e^{-2x}}{4} - \frac{4c}{4}$$

$$\frac{1}{y} = \frac{2xe^{-2x} + e^{-2x} - 4c}{4}$$

$$y = \frac{4}{2xe^{-2x} + e^{-2x} - 4c}$$

b) $\frac{dy}{dx} + \frac{xy}{1+x^2} = \sqrt{1+x^2}$

$$\frac{dy}{dx} + \left(\frac{x}{1+x^2}\right)y = \sqrt{1+x^2}$$

$$\frac{dy}{dx} + P(x).y = Q(x)$$

$P(x) = \left(\frac{x}{1+x^2}\right)$ $\quad$ $Q(x) = \sqrt{1+x^2}$

$$V(x) = e^{\int P(x)dx}$$

$$= e^{\int \frac{x}{1+x^2}dx}$$

$$= e^{\frac{1}{2}\int \frac{2x}{1+x^2}dx}$$

$$= e^{\frac{1}{2}\ln(1+x^2)}$$

$$= e^{\ln(1+x^2)^{\frac{1}{2}}}$$

$$a^{\log_a x} = x$$

$$= (1+x^2)^{\frac{1}{2}}$$

$$V(x).y = \int V(x)Q(x)\,dx$$

$$(1+x^2)^{\frac{1}{2}}.y = \int (1+x^2)^{\frac{1}{2}}.\sqrt{1+x^2}\,dx$$

$$(1+x^2)^{\frac{1}{2}}.y = \int 1+x^2\,dx$$

$$(1+x^2)^{\frac{1}{2}}.y = x + \frac{x^3}{3} + C$$

$When\ x = 0, y = 1$

$$(1+0^2)^{\frac{1}{2}}.(1) = 0 + \frac{0^3}{3} + C$$

$$C = 1$$

$$\therefore The\ particular\ solution: y\sqrt{1+x^2} = x + \frac{x^3}{3} + 1$$

10. Given the curve $y^2 = x$ and the line $y = -2x + 1$.

a) Determine the points of intersection between the curve and the line.

b) Sketch the curve and the line on the same axes. Shade the region R bounded by the curve and the line. Label the points of intersection.

c) Find the area of the region R.

d) Calculate the volume of the solid generated when the region R is rotated 2π radian about the y-axis.

SOLUTION

a) $y^2 = x$ and the line $y = -2x + 1$.

$y^2 = x$ .... (1)

$y = -2x + 1$ .... (2)

$Substitute\ (1)\ into\ (2)$

$y = -2y^2 + 1$

$2y^2 + y - 1 = 0$

$(2y - 1)(y + 1) = 0$

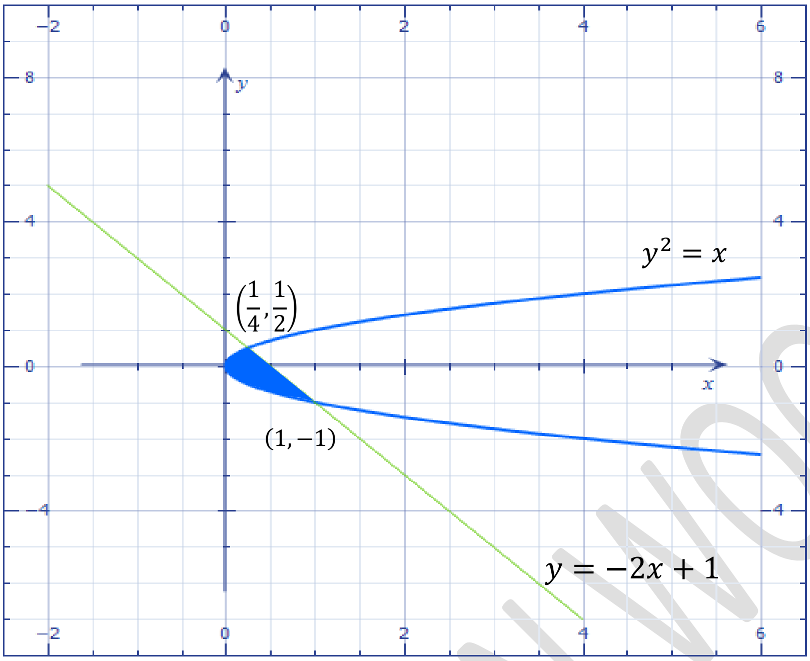
$y = \frac{1}{2}$ or $y = -1$

$x = y^2$

$x = \frac{1}{4}$ $x = 1$

$\therefore The\ intersection\ points: (1, -1), \left(\frac{1}{4}, \frac{1}{2}\right)$

(b)



$$(c)\ Area = \int_{-1}^{\frac{1}{2}} \left(\frac{y-1}{-2}\right) - (y^2)dy$$

$$= \int_{-1}^{\frac{1}{2}} \left(\frac{y-1+2y^2}{-2}\right) dy$$

$$= \frac{1}{-2}\int_{-1}^{\frac{1}{2}} 2y^2 + y - 1\, dy$$

$$= \frac{1}{-2}\left[\frac{2y^3}{3} + \frac{y^2}{2} - y\right]_{-1}^{\frac{1}{2}}$$

$$= \frac{1}{-2}\left\{\left[\frac{2\left(\frac{1}{2}\right)^3}{3} + \frac{\left(\frac{1}{2}\right)^2}{2} - \left(\frac{1}{2}\right)\right] - \left[\frac{2(-1)^3}{3} + \frac{(-1)^2}{2} - (-1)\right]\right\}$$

$$= \frac{1}{-2}\left\{\left[\frac{1}{12} + \frac{1}{8} - \frac{1}{2}\right] - \left[\frac{-2}{3} + \frac{1}{2} + 1\right]\right\}$$

$$= \frac{1}{-2}\left\{\left[\frac{2+3-12}{24}\right] - \left[\frac{-4}{6} + \frac{3}{6} + \frac{6}{6}\right]\right\}$$

$$= \frac{1}{-2}\left(\frac{-7}{24} - \frac{5}{6}\right)$$

$$= \frac{1}{-2}\left(\frac{-7}{24} - \frac{20}{24}\right)$$

$$= \frac{1}{-2}\left(\frac{-27}{24}\right)$$

$$= \frac{1}{-2}\left(\frac{-9}{8}\right)$$

$$= \frac{9}{16}\ unit^2$$

(d) $Volume = \pi \int_{-1}^{\frac{1}{2}} \left(\frac{y-1}{-2}\right)^2 - (y^2)^2 dy$

$$= \pi \int_{-1}^{\frac{1}{2}} \frac{y^2 - 2y + 1}{4} - y^4 dy$$

$$= \pi \int_{-1}^{\frac{1}{2}} \frac{y^2 - 2y + 1 - 4y^4}{4} dy$$

$$= \frac{\pi}{4} \int_{-1}^{\frac{1}{2}} -4y^4 + y^2 - 2y + 1\, dy$$

$$= \frac{\pi}{4}\left[\frac{-4y^5}{5} + \frac{y^3}{3} - \frac{2y^2}{2} + y\right]_{-1}^{\frac{1}{2}}$$

$$= \frac{\pi}{4}\left[\frac{-4y^5}{5} + \frac{y^3}{3} - y^2 + y\right]_{-1}^{\frac{1}{2}}$$

$$= \frac{\pi}{4}\left\{\left(\frac{-4\left(\frac{1}{2}\right)^5}{5} + \frac{\left(\frac{1}{2}\right)^3}{3} - \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)\right) - \left(\frac{-4(-1)^5}{5} + \frac{(-1)^3}{3} - (-1)^2 + (-1)\right)\right\}$$

$$= \frac{\pi}{4}\left\{\left(\frac{-1}{40}+\frac{1}{24}-\frac{1}{4}+\frac{1}{2}\right)-\left(\frac{4}{5}-\frac{1}{3}-1-1\right)\right\}$$

$$= \frac{\pi}{4}\left\{\left(\frac{-24+40-240+480}{960}\right)-\left(\frac{12-5-15-15}{15}\right)\right\}$$

$$= \frac{\pi}{4}\left\{\left(\frac{256}{960}\right)+\left(\frac{23}{15}\right)\right\}$$

$$= \frac{\pi}{4}\left(\frac{256+1472}{960}\right)$$

$$= \frac{\pi}{4}\left(\frac{1782}{960}\right)$$

$$= \frac{9}{20}\pi\, unit^3$$