KEMENTERIAN PENDIDIKAN MALAYSIA

QS 025/2

Matriculation Programme Examination

Semester II

Session 2016/2017

Chow Choon Wooi

1. Given $P(A) = 0.35$ and $P(B) = 0.45$. Calculate

   a. $P(A \cup B)$ if events A and B are mutually exclusive.

   b. $P(A \cap B')$ if events A and B are independent.

SOLUTION

a) $P(A) = 0.35, P(B) = 0.45$

Events A and B are mutually exclusive → $P(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B)$

$= 0.35 + 0.45$

$= 0.8$

b) $P(A) = 0.35, P(B) = 0.45$

Events A and B are independent → $P(A \cap B) = P(A).P(B)$

$P(A \cap B) = P(A).P(B)$

$= (0.35).(0.45)$

$= 0.1575$

$P(A \cap B') = P(A) - P(A \cap B)$

$= 0.35 - 0.1575$

$= 0.1925$

De Morgan Rule

$P(A' \cup B') = P(A \cap B)'$

$P(A' \cap B') = P(A \cup B)'$

$P(A \cap B') = P(A) - P(A \cap B)$

ALTERNATIVE 1(b)

$P(A) = 0.35, P(B) = 0.45$

$P(B') = 1 - P(B)$

$= 1 - 0.45$

$= 0.55$

$P(A \cap B') = P(A).P(B')$

$= (0.35).(0.55)$

$= 0.1925$

2. The mean survival times (weeks), $\bar{x}$, of a sample of 20 animals in a clinical trial is 28 with summary statistics $\sum x^2 = 18000$.

   a. Find the standard deviation correct to three decimal places.

   b. It is known that the median is 26, compute Pearson's Coefficient of Skewness. Comment on your answer.

SOLUTION

$$n = 20,\ \bar{x} = 28,\quad \sum x^2 = 18000$$

$$\bar{x} = \frac{\sum x}{n}$$

$$28 = \frac{\sum x}{20}$$

$$\sum x = 20(28)$$

$$= 560$$

a) Standard deviation

$$s = \sqrt{\frac{\sum x^2 - \frac{1}{n}(\sum x)^2}{n-1}}$$

$$s = \sqrt{\frac{18000 - \frac{1}{20}(560)^2}{20-1}}$$

$$= \sqrt{\frac{18000 - \frac{1}{20}(560)^2}{20-1}}$$

$$= \sqrt{\frac{2320}{19}}$$

$$= 11.050$$

b) Median = 26

Pearson's Coefficient of Skewness

$$S_k = \frac{3(mean - median)}{Standard\ Deviation}$$

$$S_k = \frac{3(\bar{x} - median)}{s}$$

$$= \frac{3(28 - 26)}{11.050}$$

$$= 0.543$$

$\boldsymbol{Comment: Data\ is\ skewed\ to\ the\ right}$

3. The table below shows the classification of 200 shirts based on sizes and colours.

	Small	Medium	Large
White	40	35	5
Blue	10	30	15
Black	25	20	20

A shirt is selected randomly. Find the probability that the shirt is

a. Small ini size.

b. Either blue or white.

c. Medium size given that it is blue.

SOLUTION

a)

	Small	Medium	Large	Total
White	40	35	5	***80***
Blue	10	30	15	***55***
Black	25	20	20	***65***
Total	***65***	***85***	***40***	***200***

$$P(Small) = \frac{40 + 10 + 25}{200}$$

$$= \frac{3}{8}$$

$$= 0.375$$

	Small	Medium	Large	Total
White	40	35	5	***80***
Blue	10	30	15	***55***
Black	25	20	20	***65***
Total	***65***	***85***	***40***	***200***

b) $P(Blue \cup White) = \frac{10+30+15}{200} + \frac{40+35+5}{200}$

$$= \frac{135}{200}$$

$$= \frac{27}{40}$$

$$= 0.675$$

	Small	Medium	Large	Total
White	40	35	5	***80***
Blue	10	30	15	***55***
Black	25	20	20	***65***
Total	***65***	***85***	***40***	***200***

c) $P(Medium \mid Blue) = \frac{P(Medium \cap Blue)}{P(Blue)}$

$$= \frac{\frac{30}{200}}{\frac{55}{200}}$$

$$= \frac{30}{55}$$

$$= \frac{6}{11}$$

$$= 0.545$$

4. For every class of 40 students, on average there are 4 of them are left-handed. Find the probability that

   a. Exactly 5 students are left-handed in any class.

   b. Between 4 and 17 students are left-handed in any two classes.

SOLUTION

a) $\lambda = 4\ (per\ 40\ students)$

$X \sim P_o(4)$

$P(X = 5) = \dfrac{e^{-4}.4^5}{5!}$

$P(X = x) = \dfrac{e^{-\lambda}.\lambda^x}{x!}$

$= 0.1563$

b) $\lambda = 8\ (per\ 80\ students)$

$X \sim P_o(8)$

$P(4 < X < 17) = P(X \geq 5) - P(X \geq 17)$

$= 0.9004 - 0.0037$

$= 0.8967$

5. The following list is the number of car thefts during the year 2013 in 11 particular cities.

110	340	210	300	660	115	135	400	180	145	265

a. Find the median

b. Draw a box-and-whisker plot to represent the data. Hence, state the shape of the distribution of the data and give your reason.

SOLUTION

110	115	135	145	180	210	265	300	340	400	660

a) $n = 11,$ $X_{\left(\frac{11+1}{2}\right)} = X_6$

Median = 210

b) $Q_1 = 135$

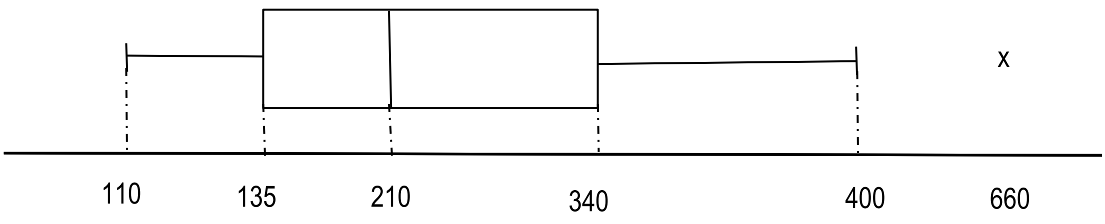
Median = 210

$Q_3 = 340$

$IQR = Q_3 - Q_1 = 340 - 135 = 205$

Lower fence = $Q_1 - 1.5\,IQR = 135 - 1.5(205) = -172.5$

Upper fence = $Q_3 + 1.5\,IQR = 340 + 1.5(205) = 647.5$



Shape of Distribution:

Positive skewness since the right box is longger than the left

6. (a) A total of 6 students can sit on 10 chairs which are arranged in a row.

i. Find the number of different ways that all the 6 students can sit.

ii. If both seats at the ends are to be seated, find the number of different ways this can be done.

iii. If 2 particular students do not sit next to each other, find the number of different ways that all 6 students can sit.

(b) A committee consisting of 2 males and 3 females is to be formed from 5 males and 7 females. Find the number of different ways if

i. A particular female must be in the committee.

ii. 2 particular males cannot be in the committee.

SOLUTION

ai) ${}^{10}P_6 = 151200$

aii) ${}^{6}P_2 \text{ x } {}^{8}P_4 = 50400$

aiii) ${}^{10}P_6 - ({}^{9}P_5)(2!)=120960$

bi) The total number of possible selection = ${}^{5}C_2 \text{ x } {}^{6}P_2 = 150$

bii) The total number of possible selection = ${}^{3}C_2 \text{ x } {}^{7}P_3 = 105$

7. The number of times, X, a certain statistics book is borrowed from a library per semester is modeled as probability distribution function below

$$P(X = x) = \begin{cases} k(7 - 2x), & x = 0, 1, 2, 3 \\ 0, & otherwise \end{cases}$$

With k as a constant. Find k.

Hence,

a. Construct a probability distribution table for X.

b. Find $P(X \leq 2)$

c. Calculate $E(2X + 3)$

d. Find $Var(2X + 3)$

SOLUTION

$$P(X = x) = \begin{cases} k(7 - 2x), & x = 0, 1, 2, 3 \\ 0, & otherwise \end{cases}$$

x	0	1	2	3
$P(X = x)$	$7k$	$5k$	$3k$	k

Probability Distribution Function ➔ $\sum P(X = x) = 1$

$$\sum P(X = x) = 1$$

$$7k + 5k + 3k + k = 1$$

$$16k = 1$$

$$k = \frac{1}{16}$$

a)

x	0	1	2	3
$P(X=x)$	$\frac{7}{16}$	$\frac{5}{16}$	$\frac{3}{16}$	$\frac{1}{16}$

b) $P(X \leq 2) = P(X=2) + P(X=1) + P(X=0)$

$$= \frac{3}{16} + \frac{5}{16} + \frac{7}{16}$$

$$= \frac{15}{16}$$

c) $E(2X+3) = 2E(x) + 3$

$$E(x) = 0\left(\frac{7}{16}\right) + 1\left(\frac{5}{16}\right) + 2\left(\frac{3}{16}\right) + 3\left(\frac{1}{16}\right)$$

$$= \frac{14}{16}$$

$$= \frac{7}{8}$$

$$E(2X+3) = 2E(x) + 3$$

$$= 2\left(\frac{7}{8}\right) + 3$$

$$= \frac{19}{4}$$

Properties of expectation

a) $E(a) = a$
b) $E(aX) = aE(x)$
c) $E(aX+b) = aE(x) + b$

d) $Var(2X+3) = 4Var(X)$

$$Var(X) = E(x^2) - [E(x)]^2$$

$$E(x) = \frac{7}{8}$$

$$E(x^2) = 0^2\left(\frac{7}{16}\right) + 1^2\left(\frac{5}{16}\right) + 2^2\left(\frac{3}{16}\right) + 3^2\left(\frac{1}{16}\right)$$

$$= \frac{26}{16}$$

$$= \frac{13}{8}$$

$$Var(X) = E(x^2) - [E(x)]^2$$

$$= \frac{13}{8} - \left[\frac{7}{8}\right]^2$$

$$= \frac{13}{8} - \left[\frac{7}{8}\right]^2$$

$$= \frac{55}{64}$$

$$Var(2X + 3) = 4Var(X)$$

$$= 4\left(\frac{55}{64}\right)$$

$$= \frac{55}{16}$$

$$= 3.4375$$

8. Let the probability density function of a continuous random variable X be defined by

$$f(x) = \begin{cases} \dfrac{x^2}{18}, & -c < x < c \\ 0, & otherwise \end{cases}$$

a. Show that c = 3.

b. Find the cumulative distribution function of X.

c. Hence, find

i. $P(0 \leq X \leq 2)$.

ii. $the\ median\ of\ X.$

SOLUTION

$$f(x) = \begin{cases} \dfrac{x^2}{18}, & -c < x < c \\ 0, & otherwise \end{cases}$$

a) $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{-c} 0dx + \int_{-c}^{c} \frac{x^2}{18}dx + \int_{c}^{\infty} 0dx = 1$$

$$\left[\frac{x^3}{54}\right]_{-c}^{c} = 1$$

$$\left[\frac{c^3}{54}\right] - \left[\frac{-c^3}{54}\right] = 1$$

$$\frac{c^3}{54} + \frac{c^3}{54} = 1$$

$$\frac{2c^3}{54} = 1$$

$$2c^3 = 54$$

$$c^3 = 27$$

$$c = 3$$

b) Cumulative Distribution Function Of X

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$f(x) = \begin{cases} \dfrac{x^2}{18}, & -3 < x < 3 \\ 0, & otherwise \end{cases}$$

x<-3	$F(x) = \int_{-\infty}^{x} 0\,dt$ $= 0$
$-3 \le x < 3$	$F(x) = \int_{-\infty}^{-3} 0\,dt + \int_{-3}^{x} \frac{t^2}{18}dt$ $= 0 + \left[\frac{t^3}{54}\right]_{-3}^{x}$ $= \left[\frac{x^3}{54}\right] - \left[\frac{-3^3}{54}\right]$ $= \frac{x^3}{54} + \frac{27}{54}$ $= \frac{x^3}{54} + \frac{1}{2}$
$x \ge 3$	$F(x) = 1$

$$F(x) = \begin{cases} 0, & x < -3 \\ \dfrac{x^3}{54} + \dfrac{1}{2}, & -3 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

c) i) $P(0 \le X \le 2) = F(2) - F(0)$

$$= \left[\frac{2^3}{54} + \frac{1}{2}\right] - \left[\frac{0^3}{54} + \frac{1}{2}\right]$$

$$= \frac{4}{27}$$

c) ii) $F(m) = 0.5$

$$\frac{m^3}{54} + \frac{1}{2} = \frac{1}{2}$$

$$\frac{m^3}{54} = 0$$

$$m^3 = 0$$

$$m = 0$$

9. The amount of grains packed in a sack is normally distributed with mean weight μ and standard deviation 6 kg. Given $P(X < 24) = 0.1587$. The sack is seperated from the others if it weighs less that 25kg.

   a. Find the value of μ.

   b. Hence,

      i. Find the probability that a randomly chosen sack has weights of more than 33 kg.

      ii. Find the probability that a randomly chosen sack will be seperated.

   c. A total of 5 sacks are chosen at random, find the probability that

      i. All the sacks are to be separated.

      ii. At least 4 of the sacks are to be separated.

SOLUTION

a) $\sigma = 6$

$P(X < 24) = 0.1587$

$$Z = \frac{X-\mu}{\sigma}$$

$P\left(Z < \dfrac{24-\mu}{6}\right) = 0.1587$

$\frac{24-\mu}{6} = -1$ (From Statiscal table)

$24 - \mu = -6$

$\mu = 30$

b) i) $P(X > 33) = P\left(Z > \frac{33-30}{6}\right)$

$= P(Z > 0.5)$

$= 0.3085$

b) ii) $P(X < 25) = P\left(Z < \frac{25-30}{6}\right)$

$= P(Z < -0.83)$

$= 0.2033$

$X \sim B(n, p)$

$P(X = x) = {}^{n}C_{x}(p)^{x}(q)^{n-x}$

c) $X \sim B(5, 0.2033)$

i. $P(X = 5) = {}^{5}C_{5}(0.2033)^{5}(0.7967)^{0}$

$= 0.0003473$

ii. $P(X \geq 4) = {}^{5}C_{4}(0.2033)^{4}(0.7967)^{1} + {}^{5}C_{5}(0.2033)^{5}(0.7967)^{0}$

$= 0.00715$

10. A games is conducted by tossing a biased coin 3 times. The coin has probability $P(H) = \frac{1}{3}$ and $P(T) = \frac{2}{3}$, where the event in obtaining head is H and the event in obtaining tail is T.

a. Construct a tree diagram and hence, show that the probability of getting one head is $\frac{12}{27}$.

b. Let X be the number of heads that appears, find the probability distribution of X.

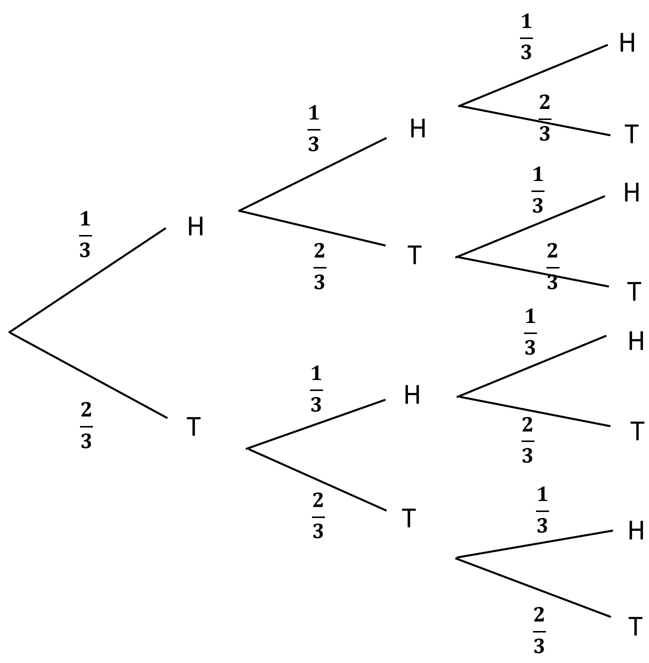
c. Suppose a player wins RM2 each time a tail appears. If Y is the profit,

   i. Find the probability distribution of Y.

   ii. Calculate $E(Y)$ and $Var(Y)$.

SOLUTION

a)



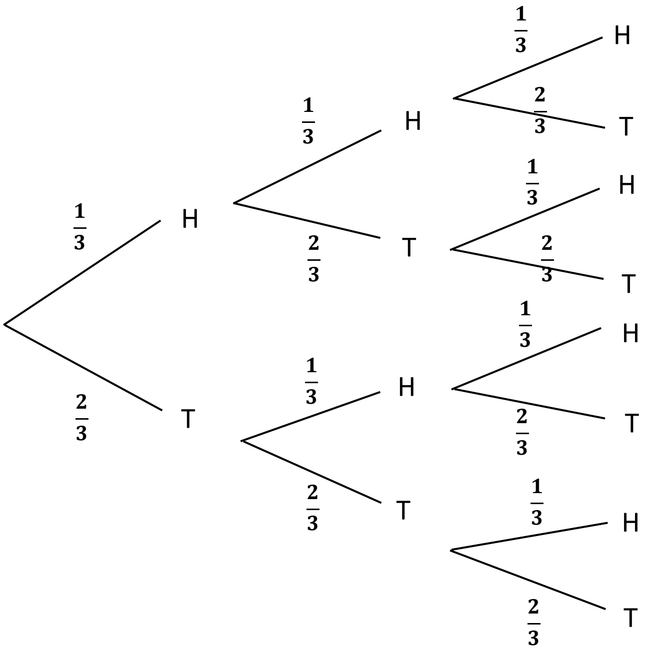
Outcomes	Probabilities
H,H,H	$\frac{1}{27}$
H,H,T	$\frac{2}{27}$
H,T,H	$\frac{2}{27}$
H,T,T	$\frac{4}{27}$
T,H,H	$\frac{2}{27}$
T,H,T	$\frac{4}{27}$
T,T,H	$\frac{4}{27}$
T,T,T	$\frac{8}{27}$

$$P(1\ Head) = P(H \cap T \cap T) + P(T \cap H \cap T) + P(T \cap T \cap H)$$

$$= \frac{4}{27} + \frac{4}{27} + \frac{4}{27}$$

$$= \frac{12}{27}$$

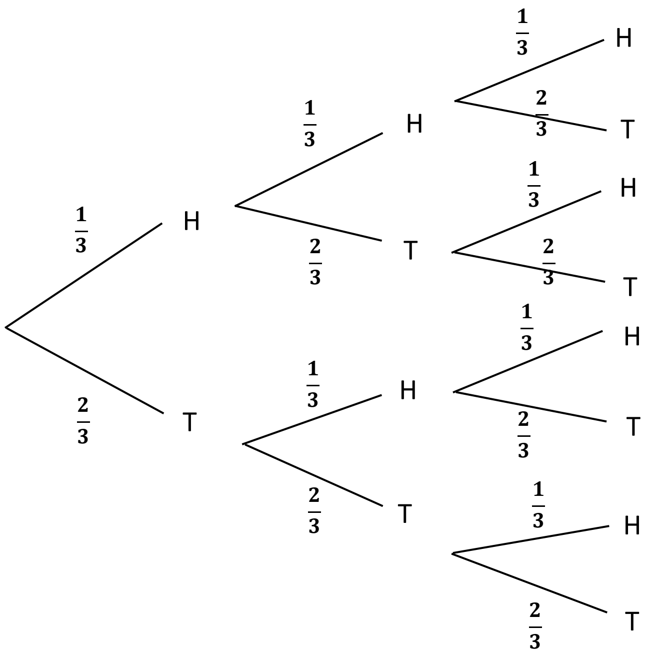
b) $X - Number\ of\ heads\ that\ appears.$



Outcomes	Probabilities	X
H.H.H	$\frac{1}{27}$	3
H.H.T	$\frac{2}{27}$	2
H.T.H	$\frac{2}{27}$	2
H,T,T	$\frac{4}{27}$	1
T.H.H	$\frac{2}{27}$	2
T,H,T	$\frac{4}{27}$	1
T,T,H	$\frac{4}{27}$	1
T.T.T	$\frac{8}{27}$	0

x	0	1	2	3
$P(X = x)$	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$

c) i) $Y - Profit$



Outcomes	Probabilities	Y
H,H,H	$\frac{1}{27}$	0T= 0
H,H,T	$\frac{2}{27}$	1T= 2
H,T,H	$\frac{2}{27}$	1T= 2
H,T,T	$\frac{4}{27}$	2T= 4
T,H,H	$\frac{2}{27}$	1T= 2
T,H,T	$\frac{4}{27}$	2T= 4
T,T,H	$\frac{4}{27}$	2T= 4
T,T,T	$\frac{8}{27}$	3T= 6

y	0	2	4	6
$P(Y = y)$	$\frac{1}{27}$	$\frac{6}{27}$	$\frac{12}{27}$	$\frac{8}{27}$

c) ii) $E(Y) = 0\left(\frac{1}{27}\right) + 2\left(\frac{6}{27}\right) + 4\left(\frac{12}{27}\right) + 6\left(\frac{8}{27}\right)$

$$= 0 + \left(\frac{12}{27}\right) + 4\left(\frac{48}{27}\right) + \left(\frac{48}{27}\right)$$

$$= 4$$

$$E(Y^2) = 0^2\left(\frac{1}{27}\right) + 2^2\left(\frac{6}{27}\right) + 4^2\left(\frac{12}{27}\right) + 6^2\left(\frac{8}{27}\right)$$

$$= 0 + \left(\frac{24}{27}\right) + \left(\frac{192}{27}\right) + \left(\frac{288}{27}\right)$$

$$= \frac{504}{27}$$

$$V(Y) = E(Y^2) - [E(Y)]^2$$

$$= \frac{504}{27} - [4]^2$$

$$= 2.667$$