KEMENTERIAN
PENDIDIKAN
MALAYSIA

QS 015/2

Matriculation Programme Examination

Semester I

Session 2015/2016

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1. Express $\frac{5x^2+4x+4}{(x^2-4)(x+2)}$ in the form of partial fractions.

2. Evaluate the following(if exist):

a) $\lim_{x\to 2} \frac{x^2+4x-12}{|x-2|}$

b) $\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}$

3. Find the derivative of the following functions:

a) $f(x) = cot\sqrt{4x^2+1}$

b) $f(x) = e^{2x}\ln(3x+4)$

4. Given $cosec^2x - \cot x = 3$, show that $cot^2x - \cot x - 2 = 0$. Hence, solve the equation $cosec^2x - cotx = 3$ for $0 \le x \le \pi$.

5. A polynomial $P(x) = 2x^4 + ax^3 + bx^2 - 17x + c$ where $a, b\ and\ c$ are constants, has factors $(x+2)$ and $(x-1)$. When $P(x)$ is divided by $(x+1)$, the remainder is 8. Find the values of $a, b\ and\ c$. Hence, factorize $P(x)$ completely and state its zeroes.

6. (a) Evaluate $\lim_{x\to\infty} \frac{\sqrt{2x^2+3x}}{5x+1}$.

(b) Given $f(x) = \begin{cases} 5-px & , \quad -2 < x \le -1 \\ x^2+px+q & , \quad -1 < x \le 2 \\ \frac{x^2-4}{x-2} & , \quad x > 2 \end{cases}$

(i) Find the values of p and q if $f(x)$ is continuous for all real values of x.

(ii) Sketch the graph of $f(x)$ using the values p and q obtained in part (i).

7. A curve is given by the parametric equations $x = t - \frac{1}{t}, y = t + \frac{1}{t}$.

a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terns of t.

b) Obtain the coordinates of the stationary points of the curve and determine the nature of the points.

8. (a) If $y^2 - 2y\sqrt{(1+x^2)} + x^2 = 0,$ show that $\frac{dy}{dx} = \frac{x}{\sqrt{(1+x^2)}}$.

(b) Water is running at a steady rate of $36\pi\ cm^3s^{-1}$ into a right inverted circular cone with a semi-vertical angle of 45°.

(i) Find the rate of increasing in water depth when the water level is 3 cm.

(ii) Find the time taken when the depth of the water is 18cm.

9. (a) Determine the values of R and α, where $R > 0$ and $0° < \alpha < 90°$ so that $3\sin\theta - 4\cos\theta = Rsin(\theta - \alpha)$.

(b) Hence, solve the equation $3\sin\theta - 4\cos\theta = 2$ for $0° \le \theta \le 360°$.

(c) From the answer obtained in part (b), find the value of θ for $0° \le \theta \le 360°$ so that $f(\theta) = \frac{1}{3sin\theta - 4\,cos\theta + 15}$ is minimum. Hence, find the minimum value of f.

10. (a) Find the value of k if the slope of the curve $x^3 + kx^2y - 2y^2 = 0$ at the point (-1,1) is -3.

(b) Given $y = \frac{\sin x}{1+\cos x}$.

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x.

(ii) Hence, show that $\frac{d^3y}{dx^3} - y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0$.

1. Express $\frac{5x^2+4x+4}{(x^2-4)(x+2)}$ in the form of partial fractions.

SOLUTION

$$\frac{5x^2+4x+4}{(x^2-4)(x+2)} = \frac{5x^2+4x+4}{(x-2)(x+2)(x+2)}$$

$$= \frac{5x^2+4x+4}{(x-2)(x+2)^2}$$

$$= \frac{A}{(x-2)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$$

$$= \frac{A(x+2)^2 + B(x+2)(x-2) + C(x-2)}{(x-2)(x+2)^2}$$

$$\frac{5x^2+4x+4}{(x-2)(x+2)^2} = \frac{A(x+2)^2 + B(x+2)(x-2) + C(x-2)}{(x-2)(x+2)^2}$$

$$5x^2+4x+4 = A(x+2)^2 + B(x+2)(x-2) + C(x-2)$$

$When\ x = 2:$

$$5(2)^2 + 4(2) + 4 = A(2+2)^2$$

$$32 = 16A$$

$$A = 2$$

$When\ x = -2:$

$$5(-2)^2 + 4(-2) + 4 = C[(-2)-2]$$

$$16 = -4C$$

$$C = -4$$

$When\ x = 0, A = 2, C = -4:$

$$5(0)^2 + 4(0) + 4 = 2[(0)+2]^2 + B(0+2)(0-2) + (-4)(0-2)$$

$$4 = 2(4) - 4B + 8$$

$4B = 12$

$B = 3$

$$\frac{5x^2 + 4x + 4}{(x^2 - 4)(x + 2)} = \frac{2}{(x - 2)} + \frac{3}{(x + 2)} - \frac{4}{(x + 2)^2}$$

2. Evaluate the following(if exist):

a) $\lim_{x \to 2} \frac{x^2+4x-12}{|x-2|}$

b) $\lim_{x \to 1} \frac{1-\sqrt{x}}{1-x}$

SOLUTION

(a)

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{|x - 2|} = \lim_{x \to 2} \frac{(x+6)(x-2)}{|x-2|}$$

$$|x - 2| = \begin{cases} (x-2), & x \geq 2 \\ -(x-2), & x < -2 \end{cases}$$

$$\frac{(x+6)(x-2)}{|x-2|} = \begin{cases} \dfrac{(x+6)(x-2)}{x-2}, & x \geq 2 \\ \dfrac{(x+6)(x-2)}{-(x-2)}, & x < -2 \end{cases}$$

$$= \begin{cases} x+6, & x \geq 2 \\ -(x+6), & x < -2 \end{cases}$$

$$\lim_{x \to 2^-} \frac{(x+6)(x-2)}{|x-2|} = \lim_{x \to 2^-} -(x+6)$$

$$= -(2+6)$$

$$= -8$$

$$\lim_{x \to 2^+} \frac{(x+6)(x-2)}{|x-2|} = \lim_{x \to 2^+} (x+6)$$

$$= (2+6)$$

$$= 8$$

$$Since \lim_{x \to 2^-} \frac{(x+6)(x-2)}{|x-2|}$$

$$\neq \lim_{x \to 2^+} \frac{(x+6)(x-2)}{|x-2|}, therefore \lim_{x \to 2} \frac{x^2+4x-12}{|x-2|} \ does\ not\ exits.$$

(b)

$$\lim_{x\to 1}\frac{1-\sqrt{x}}{1-x}=\lim_{x\to 1}\frac{1-\sqrt{x}}{1^2-\left(\sqrt{x}\right)^2}$$

$$=\lim_{x\to 1}\frac{1-\sqrt{x}}{\left(1+\sqrt{x}\right)\left(1-\sqrt{x}\right)}$$

$a^2+b^2=(a+b)(a-b)$

$$=\lim_{x\to 1}\frac{1}{\left(1+\sqrt{x}\right)}$$

$$=\frac{1}{\left(1+\sqrt{1}\right)}$$

$$=\frac{1}{2}$$

3. Find the derivative of the following functions:

a) $f(x) = cot\sqrt{4x^2 + 1}$

b) $f(x) = e^{2x} \ln(3x + 4)$

SOLUTION

(a)

$$f(x) = cot\sqrt{4x^2 + 1}$$

$$f(x) = cot(4x^2 + 1)^{\frac{1}{2}}$$

$$f'^{(x)} = -cosec^2 \, (4x^2 + 1)^{\frac{1}{2}} \, \frac{d}{dx}(4x^2 + 1)^{\frac{1}{2}}$$

$$= -cosec^2 \, (4x^2 + 1)^{\frac{1}{2}} \left(\frac{1}{2}\right)(4x^2 + 1)^{-\frac{1}{2}} \frac{d}{dx}(4x^2 + 1)$$

$$= -cosec^2 \, (4x^2 + 1)^{\frac{1}{2}} \left(\frac{1}{2}\right)(4x^2 + 1)^{-\frac{1}{2}}(8x)$$

$$= -\left(\frac{1}{2}\right)(8x)(4x^2 + 1)^{-\frac{1}{2}} \, cosec^2 \, (4x^2 + 1)^{\frac{1}{2}}$$

$$= -\frac{(4x)}{(4x^2 + 1)^{\frac{1}{2}}} \, cosec^2 \, (4x^2 + 1)^{\frac{1}{2}}$$

$$= -\frac{4x \, cosec^2\sqrt{(4x^2 + 1)}}{\sqrt{(4x^2 + 1)}}$$

(b)

$f(x) = e^{2x}\ln(3x+4)$

$u = e^{2x}$ $\qquad\qquad$ $v = \ln(3x+4)$

$u' = e^{2x}\frac{d}{dx}(2x)$ $\qquad\qquad$ $v' = \frac{1}{3x+4}\frac{d}{dx}(3x+4)$

$= 2e^{2x}$ $\qquad\qquad$ $= \frac{3}{3x+4}$

$$f'^{(x)} = (e^{2x})\left(\frac{3}{3x+4}\right) + [\ln(3x+4)](2e^{2x})$$

$$= \frac{3e^{2x}}{3x+4} + 2e^{2x}\ln(3x+4)$$

$$= e^{2x}\left[\frac{3}{3x+4} + 2\ln(3x+4)\right]$$

4. Given $cosec^2x - \cot x = 3$, show that $cot^2x - \cot x - 2 = 0$. Hence, solve the equation $cosec^2x - cotx = 3$ for $0 \le x \le \pi$.

SOLUTION

$cosec^2x - \cot x = 3$

$(1 + cot^2x) - \cot x = 3$

$cot^2x - \cot x + 1 = 3$

$cot^2x - \cot x + 1 - 3 = 0$

$cot^2x - \cot x - 2 = 0$

$cosec^2x - cotx = 3\,, \qquad 0 \le x \le \pi$

$cot^2x - \cot x - 2 = 0$

Let $\boldsymbol{u = cot\ x}$

$u^2 - u - 2 = 0$

$(u - 2)(u + 1) = 0$

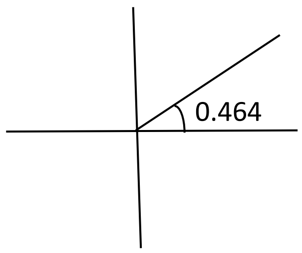
$u = 2 \qquad u = -1$

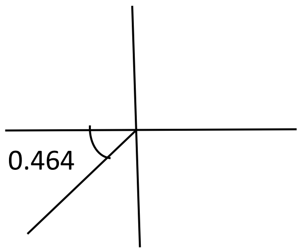
$cot\ x = 2$ or $cot\ x = -1$

$\frac{1}{\tan x} = 2$ or $\frac{1}{\tan x} = -1$

$\tan x = \frac{1}{2}$ or $\tan x = -1$

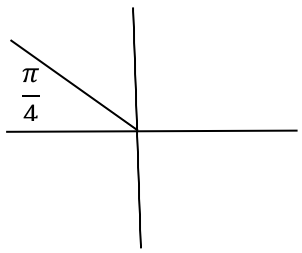
For $\tan x = \frac{1}{2}$

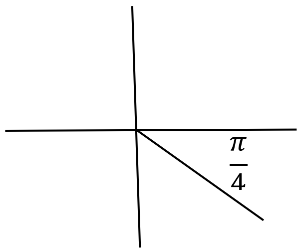




$x = 0.464$

For $\tan x = -1$





$x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\therefore \boldsymbol{x = 0.464, \frac{3\pi}{4}}$

5. A polynomial $P(x) = 2x^4 + ax^3 + bx^2 - 17x + c$ where a, b and c are constants, has factors $(x + 2)$ and $(x - 1)$. When $P(x)$ is divided by $(x + 1)$, the remainder is 8. Find the values of a, b and c. Hence, factorize $P(x)$ completely and state its zeroes.

SOLUTION

$P(-2) = 0$

$P(1) = 0$

$P(-1) = 8$

For $P(-2) = 0$

$P(x) = 2x^4 + ax^3 + bx^2 - 17x + c$

$P(-2) = 2(-2)^4 + a(-2)^3 + b(-2)^2 - 17(-2) + c$

$0 = 32 - 8a + 4b + 34 + c$

$8a - 4b - c = 66$ .... (1)

For $P(1) = 0$

$P(1) = 2(1)^4 + a(1)^3 + b(1)^2 - 17(1) + c$

$0 = 2 + a + b - 17 + c$

$a + b + c = 15$ .... (2)

For $P(-1) = 8$

$P(-1) = 2(-1)^4 + a(-1)^3 + b(-1)^2 - 17(-1) + c$

$8 = 2 - a + b + 17 + c$

$a - b - c = 11$ .... (3)

(2) + (3)

$2a = 26$

$a = 13$ .... (4)

Substitute (4) into (1)

$8(13) - 4b - c = 66$

$4b + c = 38$ .... (5)

Substitute (4) into (2)

$13 + b + c = 15$

$b + c = 2$ .... (6)

(5) – (6)

$3b = 36$

$b = 12$ .... (7)

Substitute (7) into (6)

$12 + c = 2$

$c = -10$

$\therefore a = 13, \quad b = 12, \quad c = -10$

$P(x) = 2x^4 + 13x^3 + 12x^2 - 17x - 10$

$P(-2) = 0 \rightarrow (x + 2)$ *is a factor of* $P(x)$

$P(1) = 0 \rightarrow (x - 1)$ *is a factor of* $P(x)$

$P(x) = (x + 2)(x - 1)Q(x)$

$= (x^2 + x - 2)Q(x)$

Factor Theorem

If $P(a) = 0$ *then*

$(x - a)$ *is a factor of* $P(x)$

$$
\begin{array}{r|l}
 & 2x^2 + 11x + 5 \\
x^2 + x - 2 & 2x^4 + 13x^3 + 12x^2 - 17x - 10 \\
 & 2x^4 + 2x^3 - 4x^2 \\
\hline
 & 11x^3 + 16x^2 - 17x - 10 \\
 & 11x^3 + 11x^2 - 22x \\
\hline
 & 5x^2 + 5x - 10 \\
 & 5x^2 + 5x - 10 \\
\hline
 & 0
\end{array}
$$

$P(x) = (x + 2)(x - 1)(2x^2 + 11x + 5)$

$= (x + 2)(x - 1)(2x + 1)(x + 5)$

When $P(x) = 0$

$(x + 2)(x - 1)(2x + 1)(x + 5) = 0$

$x = -2, \; x = 1, \; x = -\frac{1}{2}, \; x = -5$

6. (a) Evaluate $\lim_{x \to -\infty} \frac{\sqrt{2x^2+3x}}{5x+1}$.

(b) Given $f(x) = \begin{cases} 5 - px & , \quad -2 < x \leq -1 \\ x^2 + px + q & , \quad -1 < x \leq 2 \\ \frac{x^2-4}{x-2} & , \quad x > 2 \end{cases}$

(i) Find the values of p and q if $f(x)$ is continuous for all real values of x.

(ii) Sketch the graph of $f(x)$ using the values p and q obtained in part (i).

SOLUTION

(a)

$$\lim_{x \to -\infty} \frac{\sqrt{2x^2+3x}}{5x+1} = \lim_{x \to -\infty} \frac{\sqrt{\frac{2x^2}{x^2} + \frac{3x}{x^2}}}{\frac{5x}{x} + \frac{1}{x}}$$

$$= \lim_{x \to -\infty} \frac{\sqrt{2 + \frac{3}{x}}}{5 + \frac{1}{x}}$$

$$= \frac{-\sqrt{2+0}}{5+0}$$

$$= -\frac{\sqrt{2}}{5}$$

(b)

$$f(x) = \begin{cases} 5 - px & , \quad -2 < x \leq -1 \\ x^2 + px + q & , \quad -1 < x \leq 2 \\ \frac{x^2-4}{x-2} & , \quad x > 2 \end{cases}$$

(i) $f(x)$ is continuous at x = -1.

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$$

$$\lim_{x \to -1^-} 5 - px = \lim_{x \to -1^+} x^2 + px + q$$

$$5 - p(-1) = (-1)^2 + p(-1) + q$$

$$5 + p = 1 - p + q$$

$$2p - q = -4 \qquad \text{.........................} \ (1)$$

$f(x)$ is continuous at x = 2.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

$$\lim_{x \to 2^-} x^2 + px + q = \lim_{x \to 2^+} \frac{x^2 - 4}{x - 2}$$

$$2^2 + 2p + q = \lim_{x \to 2^+} \frac{(x+2)(x-2)}{x - 2}$$

$$4 + 2p + q = \lim_{x \to 2^+} (x + 2)$$

$$4 + 2p + q = 2 + 2$$

$$2p + q = 0 \qquad \text{.........................} \ (2)$$

(2) - (1)

$$2q = 4$$

$$q = 2$$

$$p = -1$$

$$f(x) = \begin{cases} 5 + x & , \quad -2 < x \leq -1 \\ x^2 - x + 2 & , \quad -1 < x \leq 2 \\ \dfrac{x^2 - 4}{x - 2} & , \quad x > 2 \end{cases}$$

$$= \begin{cases} 5 + x & , \quad -2 < x \leq -1 \\ x^2 - x + 2 & , \quad -1 < x \leq 2 \\ \dfrac{(x+2)(x-2)}{x - 2} & , \quad x > 2 \end{cases}$$

$$= \begin{cases} 5 + x & , \quad -2 < x \leq -1 \\ x^2 - x + 2 & , \quad -1 < x \leq 2 \\ x + 2 & , \quad x > 2 \end{cases}$$

Sketch $y = x^2 - x + 2$

1. a = 1, b = -1, c = 2
2. a > 0 (open upwards)
3. Minimum point

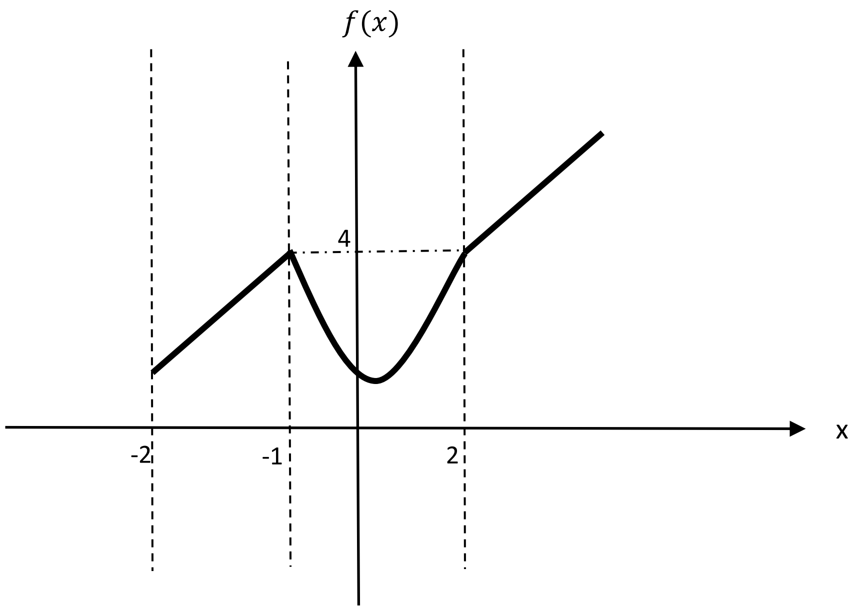
$$x = \frac{-b}{2a} = \frac{1}{2}$$

$$y = \frac{b^2 - 4ac}{-4a} = \frac{1-8}{-4} = \frac{7}{4}$$

4. Intercept

When $x = 0, \ y = 2$

(ii) Sketch the graph of $f(x)$



7. A curve is given by the parametric equations $x = t - \frac{1}{t}$, $y = t + \frac{1}{t}$.

a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terns of t.

b) Obtain the coordinates of the stationary points of the curve and determine the nature of the points.

SOLUTION

(a)

$x = t - \frac{1}{t}$ $\quad$ $y = t + \frac{1}{t}$

$x = t - t^{-1}$ $\quad$ $y = t + t^{-1}$

$\frac{dx}{dt} = 1 + t^{-2}$ $\quad$ $\frac{dy}{dt} = 1 - t^{-2}$

$\frac{dx}{dt} = 1 + \frac{1}{t^2}$ $\quad$ $\frac{dy}{dt} = 1 - \frac{1}{t^2}$

$\frac{dx}{dt} = \frac{t^2+1}{t^2}$ $\quad$ $\frac{dy}{dt} = \frac{t^2-1}{t^2}$

$$\frac{dy}{dx} = \frac{dy}{dt} . \frac{dt}{dx}$$

$$= \frac{t^2 - 1}{t^2} . \frac{t^2}{t^2 + 1}$$

$$= \frac{t^2 - 1}{t^2 + 1}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) . \frac{dt}{dx}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{t^2 - 1}{t^2 + 1}\right)$$

$u = t^2 - 1$ $\quad$ $v = t^2 + 1$

$u' = 2t$ $\quad$ $v' = 2t$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{vu' - uv'}{v^2}$$

$$= \frac{(t^2+1)(2t) - (t^2-1)(2t)}{(t^2+1)^2}$$

$$= \frac{(2t^3+2t) - (2t^3-2t)}{(t^2+1)^2}$$

$$= \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2}$$

$$= \frac{4t}{(t^2+1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) . \frac{dt}{dx}$$

$$= \frac{4t}{(t^2+1)^2} . \frac{t^2}{t^2+1}$$

$$= \frac{4t^3}{(t^2+1)^3}$$

(b)

To obtain the coordinates of the stationary points,

Let $\frac{dy}{dx} = 0$

$$\frac{t^2-1}{t^2+1} = 0$$

$$t^2 - 1 = 0$$

$$t^2 = 1$$

$$t = \pm 1$$

$When\ t = 1$

$x = t - \frac{1}{t}$ $\qquad y = t + \frac{1}{t}$

$x = 1 - \frac{1}{1}$ $\qquad y = 1 + \frac{1}{1}$

$x = 0$ $\qquad y = 2$

$When\ t = -1$

$x = t - \frac{1}{t}$ $\qquad y = t + \frac{1}{t}$

$x = -1 - \frac{1}{-1}$ $\qquad y = -1 + \frac{1}{-1}$

$x = 0$ $\qquad y = -2$

$The\ stationary\ points\ are\ (0, -2)\ and\ (0, 2)$

$When\ t = 1, and\ at\ the\ point\ (0,2)$

$$\frac{d^2y}{dx^2} = \frac{4t^3}{(t^2+1)^3}$$

$$= \frac{4(1)^3}{((1)^2+1)^3}$$

> 0 (Min)

$When\ t = -1, and\ at\ the\ point\ (0, -2)$

$$\frac{d^2y}{dx^2} = \frac{4t^3}{(t^2+1)^3}$$

$$= \frac{4(-1)^3}{((-1)^2+1)^3}$$

< 0 (Max)

$\therefore\ (0, -2) is\ a\ maximum\ point, \qquad (0,2) is\ a\ minimum\ point$

8. (a) If $y^2 - 2y\sqrt{(1+x^2)} + x^2 = 0$, show that $\frac{dy}{dx} = \frac{x}{\sqrt{(1+x^2)}}$.

(b) Water is running at a steady rate of $36\pi\ cm^3 s^{-1}$ into a right inverted circular cone with a semi-vertical angle of $45°$.

(i) Find the rate of increasing in water depth when the water level is 3 cm.

(ii) Find the time taken when the depth of the water is 18cm.

SOLUTION

(a) $y^2 - 2y\sqrt{(1+x^2)} + x^2 = 0$

$$y^2 - 2y(1+x^2)^{\frac{1}{2}} + x^2 = 0$$

$$2y\frac{dy}{dx} - \left[(2y)\left(\frac{1}{2}\right)(1+x^2)^{-\frac{1}{2}}(2x) + (1+x^2)^{\frac{1}{2}}\left(2\frac{dy}{dx}\right)\right] + 2x = 0$$

$$2y\frac{dy}{dx} - \left[\frac{2xy}{\sqrt{1+x^2}} + 2\sqrt{1+x^2}\frac{dy}{dx}\right] + 2x = 0$$

$$2y\frac{dy}{dx} - \frac{2xy}{\sqrt{1+x^2}} - 2\sqrt{1+x^2}\frac{dy}{dx} + 2x = 0$$

$$2y\frac{dy}{dx} - 2\sqrt{1+x^2}\frac{dy}{dx} = \frac{2xy}{\sqrt{1+x^2}} - 2x$$

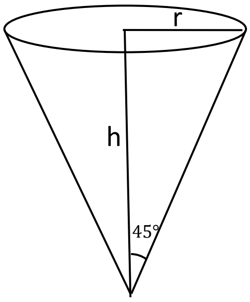
$$\frac{dy}{dx}\left(2y - 2\sqrt{1+x^2}\right) = \frac{2xy - 2x\sqrt{1+x^2}}{\sqrt{1+x^2}}$$

$$\frac{dy}{dx} = \frac{2xy - 2x\sqrt{1+x^2}}{\left(2y - 2\sqrt{1+x^2}\right)\sqrt{1+x^2}}$$

$$\frac{dy}{dx} = \frac{2x\left(y - \sqrt{1+x^2}\right)}{2\left(y - \sqrt{1+x^2}\right)\sqrt{1+x^2}}$$

$$= \frac{x}{\sqrt{1+x^2}}$$

(b) $\frac{dv}{dt} = 36\pi\ cm^3 s^{-1}$



$$\tan 45° = \frac{r}{h}$$

$$1 = \frac{r}{h}$$

$$h = r$$

(i) Find $\frac{dh}{dt}$ when h = 3 cm

$$\frac{dh}{dt} = \frac{dh}{dv} \cdot \frac{dv}{dt}$$

$$v = \frac{1}{3}\pi r^2 h$$

$$Since\ r = h$$

$$v = \frac{1}{3}\pi h^2 h$$

$$= \frac{1}{3}\pi h^3$$

$$\frac{dv}{dh} = \pi h^2$$

$$\frac{dh}{dt} = \frac{dh}{dv} \cdot \frac{dv}{dt}$$

$$= \frac{1}{(\pi h^2)} \cdot (36\pi)$$

$$= \frac{36}{h^2}$$

$$When\ h = 3\ cm$$

$$\frac{dh}{dt} = \frac{36}{3^2}$$

$$= 4cms^{-1}$$

(ii) $Find\ t\ when\ h = 18\ cm.$

$$v = \frac{1}{3}\pi h^3$$

$$= \frac{1}{3}\pi(18)^3$$

$$= 1944\pi$$

$$\frac{dv}{dt} = 36\pi$$

$$\frac{1944\pi}{t} = 36\pi$$

$$t = 54s$$

9. (a) Determine the values of R and α, where $R > 0$ and $0° < \alpha < 90°$ so that

$3\sin\theta - 4\cos\theta = R\sin(\theta - \alpha)$.

(b) Hence, solve the equation $3\sin\theta - 4\cos\theta = 2$ for $0° \le \theta \le 360°$.

(c) From the answer obtained in part (b), find the value of θ for $0° \le \theta \le 360°$ so that

$f(\theta) = \frac{1}{3sin\theta - 4\,cos\theta + 15}$ is minimum. Hence, find the minimum value of f.

SOLUTION

(a)

$3\sin\theta - 4\cos\theta = R\sin(\theta - \alpha)$.

$3\sin\theta - 4\cos\theta = R(sin\theta\cos\alpha - \cos\theta\sin\alpha)$

$3\sin\theta - 4\cos\theta = Rsin\theta\cos\alpha - \mathrm{R}\cos\theta\sin\alpha$

$\sin\theta$: $R\cos\alpha = 3$ .... (1)

$\cos\theta$: $R\sin\alpha = 4$ .... (2)

$(1)^2 + (2)^2$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2$$

$$R^2(\cos^2\alpha + \sin^2\alpha) = 25$$

$\cos^2\alpha + \sin^2\alpha = 1$

$$R^2 = 25$$

$$R = 5$$

$(2) \div (1)$

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{3}{4}$$

$$\frac{\sin\alpha}{\cos\alpha} = \frac{3}{4}$$

$$\tan\alpha = \frac{3}{4}$$

$$\alpha = 53.1°$$

$$\therefore\ 3\sin\theta - 4\cos\theta = 5\,sin(\theta - 53.1°)$$

(b)

$3\sin\theta - 4\cos\theta = 2 \qquad\qquad 0° \le \theta \le 360°$

$5\,sin(\theta - 53.1°) = 2$

$$sin(\theta - 53.1°) = \frac{2}{5}$$

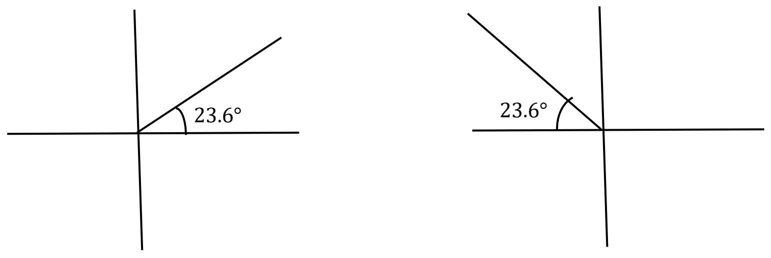
$sin(\theta - 53.1°) = 0.4$

$0° \le \theta \le 360°$

$0° - 53.1° \le \theta - 53.1° \le 360° - 53.1°$

$-53.1° \le \theta - 53.1° \le 306.9°$

$sin(\theta - 53.1°) = 0.4$



$\theta - 53.1° = 23.6°, 180° - 23.6°$

$\theta - 53.1° = 23.6°, 156.4°$

$\theta = 23.6° + 53.1°, 156.4° + 53.1°$

$\theta = 76.7°, 209.5°$

(c)

$$f(\theta) = \frac{1}{3sin\theta - 4\,cos\theta + 15}$$

$$f(\theta) = \frac{1}{5\,sin(\theta - 53.1°) + 15}$$

Since $-1 \leq sin(\theta - 53.1°) \leq 1$

For $f(\theta)$ minimum

$$sin(\theta - 53.1°) = 1$$

$$\theta - 53.1° = 90°$$

$$\theta = 143.1°$$

Therfore, the minimum value of $f(\theta)$

$$f(\theta) = \frac{1}{5\,sin(143.1° - 53.1°) + 15}$$

$$= \frac{1}{5\,sin(90°) + 15}$$

$$= \frac{1}{5\,(1) + 15}$$

$$= \frac{1}{20}$$

10. (a) Find the value of k if the slope of the curve $x^3 + kx^2y - 2y^2 = 0$ at the point (-1,1) is -3.

(b) Given $y = \frac{\sin x}{1+\cos x}$.

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x.

(ii) Hence, show that $\frac{d^3y}{dx^3} - y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0$.

SOLUTION

(a)

$$x^3 + kx^2y - 2y^2 = 0$$

$$3x^2 + \left[kx^2\frac{dy}{dx} + 2kxy\right] - 4y\frac{dy}{dx} = 0$$

$$3x^2 + kx^2\frac{dy}{dx} + 2kxy - 4y\frac{dy}{dx} = 0$$

$$kx^2\frac{dy}{dx} - 4y\frac{dy}{dx} = -2kxy - 3x^2$$

$$\frac{dy}{dx}[kx^2 - 4y] = -2kxy - 3x^2$$

$$\frac{dy}{dx} = \frac{-2kxy - 3x^2}{kx^2 - 4y}$$

At the point (-1,1) ➔ x = -1, y =1 $\frac{dy}{dx} = -3$

$$\frac{dy}{dx} = \frac{-2kxy - 3x^2}{kx^2 - 4y}$$

$$-3 = \frac{-2k(-1)(1) - 3(-1)^2}{k(-1)^2 - 4(1)}$$

$$-3=\frac{2k-3}{k-4}$$

$$-3(k-4)=2k-3$$

$$-3k+12=2k-3$$

$$12+3=2k+3k$$

$$5k=15$$

$$k=3$$

(b)

(i) $y=\frac{\sin x}{1+\cos x}$

$u=\sin x$ $v=1+\cos x$

$u'=\cos x$ $v'=-\sin x$

$$\frac{dy}{dx}=\frac{(1+\cos x)(\cos x)-(\sin x)(-\sin x)}{(1+\cos x)^2}$$

$$=\frac{\cos x+\cos^2 x+\sin^2 x}{(1+\cos x)^2}$$

$$=\frac{\cos x+1}{(1+\cos x)^2}$$

$$=\frac{1}{1+\cos x}$$

$$=(1+\cos x)^{-1}$$

$$\frac{d^2y}{dx^2}=-(1+\cos x)^{-2}\frac{d}{dx}(1+\cos x)$$

$$=-(1+\cos x)^{-2}(-\sin x)$$

$$=\frac{\sin x}{(1+\cos x)^2}$$

(ii) show that $\frac{d^3y}{dx^3} - y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0$

$$\frac{d^2y}{dx^2} = \frac{\sin x}{(1+\cos x)^2}$$

$u = \sin x \qquad v = (1+\cos x)^2$

$u' = \cos x \qquad v' = 2(1+\cos x)\frac{d}{dx}(1+\cos x)$

$$= 2(1+\cos x)(-\sin x)$$

$$= -2\sin x\,(1+\cos x)$$

$$\frac{d^3y}{dx^3} = \frac{(1+\cos x)^2(\cos x) - [-2\sin x\,(1+\cos x)](\sin x)}{[(1+\cos x)^2]^2}$$

$$= \frac{(1+\cos x)^2(\cos x) + 2\sin^2 x\,(1+\cos x)}{(1+\cos x)^4}$$

$$= \frac{(1+\cos x)[\cos x\,(1+\cos x) + 2\sin^2 x]}{(1+\cos x)^4}$$

$$= \frac{\cos x + \cos^2 x + 2\sin^2 x}{(1+\cos x)^3}$$

$$\frac{d^3y}{dx^3} - y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2$$

$$= \left(\frac{\cos x + \cos^2 x + 2\sin^2 x}{(1+\cos x)^3}\right) - y\left(\frac{\sin x}{(1+\cos x)^2}\right) - \left(\frac{1}{1+\cos x}\right)^2$$

$$= \frac{\cos x + \cos^2 x + 2\sin^2 x}{(1+\cos x)^3} - \frac{y\sin x}{(1+\cos x)^2} - \frac{1}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + 2\sin^2 x}{(1+\cos x)^3} - \frac{\left(\frac{\sin x}{1+\cos x}\right)\sin x}{(1+\cos x)^2} - \frac{1}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + 2\sin^2 x}{(1+\cos x)^3} - \frac{\sin^2 x}{(1+\cos x)^3} - \frac{1+\cos x}{(1+\cos x)^3}$$

$$= \frac{\cos x + \cos^2 x + 2\sin^2 x - \sin^2 x - (1+\cos x)}{(1+\cos x)^3}$$

$$= \frac{\cos x + \cos^2 x + 2\sin^2 x - \sin^2 x - 1 - \cos x}{(1+\cos x)^3}$$

$$= \frac{\cos^2 x + \sin^2 x - 1}{(1+\cos x)^3}$$

$$= \frac{1-1}{(1+\cos x)^3}$$

$$= 0$$