

QS 025

Mid-Semester Examination

Semester II

Session 2015/2016

1. Using integration by parts, evaluate $\int x \cos 3x \, dx$.
2. Find
 - a. $\int \frac{5-x}{\sqrt{x}} \, dx$
 - b. $\int \frac{\ln x}{x} \, dx$
3. Solve the differential equation $3x^2 \frac{dy}{dx} = (x^2 - 1)\sqrt{y}$. Express y in terms of x .
4. Show that the equation $2x^4 = 5 + x$ has a root between $x = 1$ and $x = 2$. By taking $x = 1.4$ as the first approximation, evaluate this root to three significant figures using the Newton-Raphson method.
5. The region R is enclosed by the curve $y = \sqrt{6-x}$, x -axis and $0 \leq x \leq 6$. Find
 - a. The area of the region R .
 - b. The volume of the solid generated when the region R is rotated through 2π radians about the x -axis.
6. (a) Find the general equation of the circle with centre $(3, 3)$ and radius $\sqrt{10}$. If the straight line $3x - y + 4 = 0$ is tangent to the circle at point P , find the coordinate of P . Hence, find the equation of the normal at P .

(b) Express the equation of a parabola $x^2 - 4x - 24y + 76 = 0$ in standard form. Hence, determine the coordinates of its vertex and focus.

END OF QUESTION PAPER

1. Using integration by parts, evaluate $\int x \cos 3x \, dx$.

SOLUTION

$$\int x \cos 3x \, dx$$

$$u = x$$

$$dv = \cos 3x \, dx$$

$$\frac{du}{dx} = 1$$

$$\int dv = \int \cos 3x \, dx$$

$$u = dx$$

$$v = \frac{\sin 3x}{3}$$

$$\int u \, dv = uv - \int v \, du$$

$$\int x \cos 3x \, dx = (x) \left(\frac{\sin 3x}{3} \right) - \int \frac{\sin 3x}{3} \, du$$

$$= \frac{x \sin 3x}{3} - \frac{1}{3} \int \sin 3x \, du$$

$$= \frac{x \sin 3x}{3} - \frac{1}{3} \left(-\frac{\cos 3x}{3} \right) + c$$

$$= \frac{x \sin 3x}{3} + \frac{\cos 3x}{9} + c$$

2. Find

a. $\int \frac{5-x}{\sqrt{x}} dx$

b. $\int \frac{\ln x}{x} dx$

SOLUTION

a. $\int \frac{5-x}{\sqrt{x}} dx$

$$\begin{aligned}\int \frac{5-x}{\sqrt{x}} dx &= \int \frac{5}{\sqrt{x}} dx - \int \frac{x}{\sqrt{x}} dx \\ &= 5 \int x^{-\frac{1}{2}} dx - \int x^{\frac{1}{2}} dx \\ &= 5 \left(2x^{\frac{1}{2}} \right) - \frac{2x^{\frac{3}{2}}}{3} + c \\ &= 10x^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}} + c\end{aligned}$$

a. $\int \frac{5-x}{\sqrt{x}} dx$

$$u = \sqrt{x}$$

$$u^2 = x$$

$$2u \frac{du}{dx} = 1$$

$$2u du = dx$$

$$\begin{aligned}\int \frac{5-x}{\sqrt{x}} dx &= \int \frac{5-u^2}{u} (2u du) \\ &= 2 \int 5 - u^2 du \\ &= 2 \left[5u - \frac{u^3}{3} \right] + c \\ &= 2 \left[5\sqrt{x} - \frac{\sqrt{x}^3}{3} \right] + c \\ &= 10\sqrt{x} - \frac{\sqrt{x}^3}{3} + c\end{aligned}$$

$$\mathbf{b.} \int \frac{\ln x}{x} dx$$

$$u = \ln x$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$xdu = dx$$

$$\begin{aligned} \int \frac{\ln x}{x} dx &= \int \frac{u}{x} (x du) \\ &= \int u du \\ &= \frac{u^2}{2} + C \\ &= \frac{(\ln x)^2}{2} + C \end{aligned}$$

$$\mathbf{b.} \int \frac{\ln x}{x} dx$$

$$u = \ln x \quad dv = \frac{1}{x} dx$$

$$\frac{du}{dx} = \frac{1}{x} \quad v = \int \frac{1}{x} dx$$

$$du = \frac{1}{x} dx \quad v = \ln x$$

$$\int \frac{\ln x}{x} dx = uv - \int v du$$

$$\int \frac{\ln x}{x} dx = (\ln x)(\ln x) - \int (\ln x) \left(\frac{1}{x} dx\right)$$

$$\int \frac{\ln x}{x} dx = (\ln x)^2 - \int \frac{\ln x}{x} dx$$

$$\int \frac{\ln x}{x} dx + \int \frac{\ln x}{x} dx = (\ln x)^2$$

$$2 \int \frac{\ln x}{x} dx = (\ln x)^2 + c$$

$$\int \frac{\ln x}{x} dx = \frac{(\ln x)^2}{2} + c$$

3. Solve the differential equation $3x^2 \frac{dy}{dx} = (x^2 - 1)\sqrt{y}$. Express y in terms of x .

SOLUTION

$$3x^2 \frac{dy}{dx} = (x^2 - 1)\sqrt{y}$$

$$\int \frac{1}{\sqrt{y}} dy = \int \frac{(x^2 - 1)}{3x^2} dx$$

$$\int y^{-\frac{1}{2}} dy = \int \frac{x^2}{3x^2} - \frac{1}{3x^2} dx$$

$$\int y^{-\frac{1}{2}} dy = \int \frac{1}{3} - \frac{1}{3}x^{-2} dx$$

$$\frac{y^{\frac{1}{2}}}{\frac{1}{2}} = \frac{1}{3}x - \frac{1}{3} \left(\frac{x^{-1}}{-1} \right) + c$$

$$2y^{\frac{1}{2}} = \frac{1}{3}x + \frac{1}{3x} + c$$

$$y^{\frac{1}{2}} = \frac{1}{6}x + \frac{1}{6x} + c$$

$$y = \left(\frac{1}{6}x + \frac{1}{6x} + c \right)^2$$

4. Show that the equation $2x^4 = 5 + x$ has a root between $x = 1$ and $x = 2$. By taking $x = 1.4$ as the first approximation, evaluate this root to three significant figures using the Newton-Raphson method.

SOLUTION

$$\text{Let } f(x) = 2x^4 - x - 5$$

$$f(1) = 2(1)^4 - (1) - 5 = -4 < 0$$

$$f(2) = 2(2)^4 - (2) - 5 = 25 > 0$$

Since $f(1)$ and $f(2)$ have different signs, therefore $2x^4 = 5 + x$ has a root between $x = 1$ and $x = 2$.

$$f'(x) = 8x^3 - 1$$

$$x_0 = 1.4$$

$$x_1 = 1.4 - \frac{f(1.4)}{f'(1.4)} = 1.4 - \frac{2(1.4)^4 - (1.4) - 5}{8(1.4)^3 - 1} = 1.33876$$

$$x_2 = 1.33876 - \frac{f(1.33876)}{f'(1.33876)} = 1.33876 - \frac{2(1.33876)^4 - (1.33876) - 5}{8(1.33876)^3 - 1} = 1.334046$$

$$x_3 = 1.334046 - \frac{f(1.334046)}{f'(1.334046)} = 1.33876 - \frac{2(1.334046)^4 - (1.334046) - 5}{8(1.334046)^3 - 1} = 1.33402$$

$$x_4 = 1.33402 - \frac{f(1.33402)}{f'(1.33402)} = 1.33876 - \frac{2(1.33402)^4 - (1.33402) - 5}{8(1.33402)^3 - 1} = 1.33402$$

$$x_4 = 1.33402 - \frac{f(1.33402)}{f'(1.33402)} = 1.33876 - \frac{2(1.33402)^4 - (1.33402) - 5}{8(1.33402)^3 - 1} = 1.33402$$

\therefore The root is 1.33

5. The region R is enclosed by the curve $y = \sqrt{6-x}$, x -axis and $0 \leq x \leq 6$. Find
- The area of the region R.
 - The volume of the solid generated when the region R is rotated through 2π radians about the x -axis.

SOLUTION

$$\begin{aligned} 5(a) \quad \text{Area} &= \int_0^6 \sqrt{6-x} \, dx \\ &= \int_0^6 (6-x)^{\frac{1}{2}} \, dx \\ &= \left[\frac{(6-x)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)(-1)} \right]_0^6 \\ &= -\frac{2}{3} \left[(6-x)^{\frac{3}{2}} \right]_0^6 \\ &= -\frac{2}{3} \left[(6-6)^{\frac{3}{2}} - (6-0)^{\frac{3}{2}} \right] \\ &= -\frac{2}{3} \left[0 - (6)^{\frac{3}{2}} \right] \\ &= 9.798 \quad @ \quad 4\sqrt{6} \text{ unit}^2 \end{aligned}$$

$$\begin{aligned} 5(b) \quad \text{Volume} &= \pi \int_0^6 (\sqrt{6-x})^2 \, dx \\ &= \pi \int_0^6 6-x \, dx \\ &= \pi \left[6x - \frac{x^2}{2} \right]_0^6 \\ &= \pi \left[\left(6(6) - \frac{6^2}{2} \right) - \left(6(0) - \frac{0^2}{2} \right) \right] \\ &= \pi [(36 - 18) - (0 - 0)] \\ &= 18\pi \text{ unit}^3 \end{aligned}$$

6. (a) Find the general equation of the circle with centre (3, 3) and radius $\sqrt{10}$. If the straight line $3x - y + 4 = 0$ is atangent to the circle at point P, find the coordinate of P. Hence, find the equation of the normal at P.
- (b) Express the equation of a parabola $x^2 - 4x - 24y + 76 = 0$ in standard form. Hence, determine the coordinates of its vertex and focus.

SOLUTION

6(a) Centre, $(h, k) = (3, 3)$

Radius, $r = \sqrt{10}$

$$(x - h)^2 + (y - k)^2 = (r)^2$$

$$(x - 3)^2 + (y - 3)^2 = (\sqrt{10})^2$$

$$(x^2 - 6x + 9) + (y^2 - 6y + 9) = 10$$

$$x^2 + y^2 - 6x - 6y + 18 = 10$$

$$x^2 + y^2 - 6x - 6y + 8 = 0 \quad \dots\dots\dots (1)$$

Substitute $y = 3x + 4$ into (1)

$$x^2 + (3x + 4)^2 - 6x - 6(3x + 4) + 8 = 0$$

$$x^2 + 9x^2 + 24x + 16 - 6x - 18x - 24 + 8 = 0$$

$$10x^2 = 0$$

$$x = 0$$

$$y = 3x + 4 = 3(0) + 4 = 4$$

Point P = (0,4)

$$M_T = 3$$

$$M_n = -\frac{1}{3}$$

Equations of Normal

$$y - 4 = -\frac{1}{3}(x - 0)$$

$$y = -\frac{1}{3}x + 4$$

$$x + 3y - 12 = 0$$

$$6(b) \quad x^2 - 4x - 24y + 76 = 0$$

$$\left[x^2 - 4x + \left(\frac{-4}{2}\right)^2 - \left(\frac{-4}{2}\right)^2 \right] - 24y + 76 = 0$$

$$[(x - 2)^2 - 4] - 24y + 76 = 0$$

$$(x - 2)^2 = 24y - 72$$

$$(x - 2)^2 = 24(y - 3)$$

$$(x - 2)^2 = 4(6)(y - 3)$$

$$\therefore V(h, k) = V(2, 3)$$

$$4p = 24$$

$$p = 6$$

$$F(h, k + p) = F(2, 9)$$