

**QS 025**

**Mid-Semester Examination**

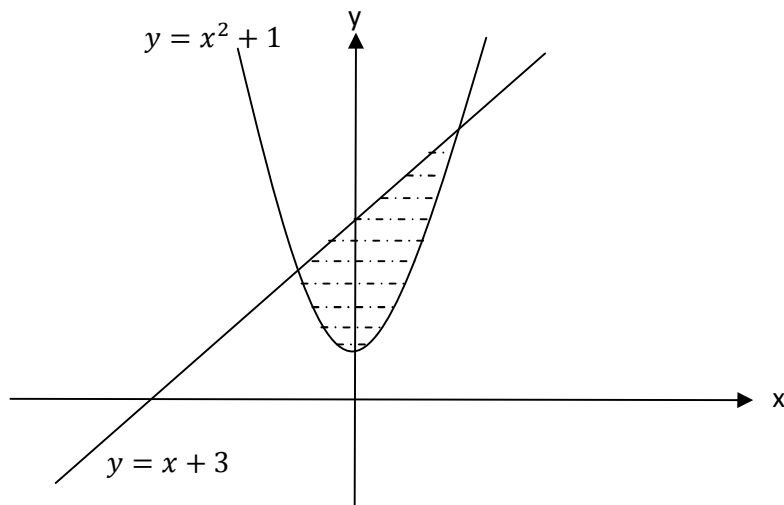
**Semester II**

**Session 2011/2012**

1. Find the area enclosed by the graph  $y = x + 3$  and  $y = x^2 + 1$ .
2. Find
  - a.  $\int (3 + e^x)(2 + e^{-x}) dx$
  - b.  $\int x \cos(2 - 4x^2) dx$  by using an appropriate substitution method.
3. Given the equation  $e^x = \frac{2}{x} + 1$ .
  - a. Show that there is a real root between 1 and 2.
  - b. By using Newton-Raphson method, find the root of the equation correct to three decimal places, taking 1.5 as the first approximation.
4. Solve the differential equation  $\frac{dy}{dx} = -2y + 3$  by using integrating factor method. Express  $y$  in terms of  $x$ .
5. Express  $\frac{3x^2 - 7x + 6}{(x-3)^2(x+1)}$  in the form of partial fraction. Hence, evaluate  $\int_1^2 \frac{3x^2 - 7x + 6}{(x-3)^2(x+1)} dx$ . Give the answer in the form of  $a + \ln b$ .
6. The Cartesian equation of an ellipse is given by  $4x^2 + 9y^2 - 16x + 36y + 16 = 0$ .
  - a. Express the Cartesian equation of the ellipse in the standard form.
  - b. Determine the coordinates of the centre, vertices and foci of the ellipse.
  - c. Sketch the graph of the ellipse.

1. Find the area enclosed by the graph  $y = x + 3$  and  $y = x^2 + 1$ .

**SOLUTION**



Intersection points:

$$x^2 + 1 = x + 3$$

$$x^2 - x - 2 = 0$$

$$(x + 1)(x - 2) = 0$$

$$x = -1 \quad \text{or} \quad x = 2$$

$$\begin{aligned} \text{Area} &= \int_{-1}^2 (x + 3) - (x^2 + 1) \, dx \\ &= \int_{-1}^2 -x^2 + x + 2 \, dx \\ &= \left[ -\frac{x^3}{3} + \frac{x^2}{2} + 2x \right]_{-1}^2 \\ &= \left[ -\frac{(2)^3}{3} + \frac{(2)^2}{2} + 2(2) \right] - \left[ -\frac{(-1)^3}{3} + \frac{(-1)^2}{2} + 2(-1) \right] \\ &= \left[ -\frac{8}{3} + \frac{4}{2} + 4 \right] - \left[ \frac{1}{3} + \frac{1}{2} - 2 \right] \\ &= \frac{9}{2} \text{ unit}^2 \end{aligned}$$

2. Find

a.  $\int (3 + e^x)(2 + e^{-x}) dx$

b.  $\int x \cos(2 - 4x^2) dx$  by using an appropriate substitution method.

**SOLUTION**

(a)

$$\begin{aligned}\int (3 + e^x)(2 + e^{-x}) dx &= \int 6 + 3e^{-x} + 2e^x + e^x e^{-x} dx \\ &= \int 6 + 3e^{-x} + 2e^x + e^0 dx \\ &= \int 7 + 3e^{-x} + 2e^x dx \\ &= 7x - 3e^{-x} + 2e^x + c\end{aligned}$$

(b)

$$\begin{aligned}\int x \cos(2 - 4x^2) dx \\ u = 2 - 4x^2 \\ \frac{du}{dx} = -8x \\ dx = -\frac{1}{8x} du \\ \int x \cos(2 - 4x^2) dx &= \int x \cos u \left(-\frac{1}{8x} du\right) \\ &= -\frac{1}{8} \int \cos u du \\ &= -\frac{1}{8} \sin u + C \\ &= -\frac{1}{8} \sin(2 - 4x^2) + C\end{aligned}$$

3. Given the equation  $e^x = \frac{2}{x} + 1$ .
- Show that there is a real root between 1 and 2.
  - By using Newton-Raphson method, find the root of the equation correct to three decimal places, taking 1.5 as the first approximation.

**SOLUTION****(a)**

$$e^x = \frac{2}{x} + 1$$

$$f(x) = e^x - \frac{2}{x} - 1$$

$$f(1) = e^1 - \frac{2}{1} - 1 = -0.2817 < 0$$

$$f(2) = e^2 - \frac{2}{2} - 1 = 5.39 > 0$$

Since  $f(1) < 0$  and  $f(2) > 0$ , therefore root lies between 1 and 2.

**(b)**

$$f(x) = e^x - \frac{2}{x} - 1$$

$$f'(x) = e^x + \frac{2}{x^2}$$

$$x_{n+1} = x_n - \frac{e^{x_n} - \frac{2}{x_n} - 1}{e^{x_n} + \frac{2}{x_n^2}}$$

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

$$x_0 = 1.5$$

$$x_1 = 1.5 - \frac{e^{1.5} - \frac{2}{1.5} - 1}{e^{1.5} + \frac{2}{(1.5)^2}} = 1.0999$$

$$x_2 = 1.0999 - \frac{e^{1.0999} - \frac{2}{1.0999} - 1}{e^{1.0999} + \frac{2}{(1.0999)^2}} = 1.0600$$

$$x_3 = 1.0600 - \frac{e^{1.0600} - \frac{2}{1.0600} - 1}{e^{1.0600} + \frac{2}{(1.0600)^2}} = 1.0601$$

$$x_4 = 1.0601 - \frac{e^{1.0601} - \frac{2}{1.0601} - 1}{e^{1.0601} + \frac{2}{(1.0601)^2}} = 1.0601$$

$\therefore$  The root is 1.060

4. Solve the differential equation  $\frac{dy}{dx} = -2y + 3$  by using integrating factor method. Express  $y$  in terms of  $x$ .

**SOLUTION**

$$\frac{dy}{dx} = -2y + 3$$

$$\frac{dy}{dx} + 2y = 3$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Integrating factor,

$$P(x) = 2; \quad Q(x) = 3$$

$$V(x) = e^{\int P(x)dx}$$

$$V(x) = e^{\int 2dx} = e^{2x}$$

$$V(x)y = \int V(x)Q(x)dx$$

$$e^{2x}y = \int e^{2x}(3)dx$$

$$ye^{2x} = 3 \int e^{2x} dx$$

$$ye^{2x} = 3 \left( \frac{e^{2x}}{2} \right) + c$$

$$ye^{2x} = \frac{3e^{2x}}{2} + c$$

$$y = \frac{3e^{2x}}{2(e^{2x})} + \frac{c}{e^{2x}}$$

$$y = \frac{3}{2} + ce^{-2x}$$

5. Express  $\frac{3x^2-7x+6}{(x-3)^2(x+1)}$  in the form of partial fraction. Hence, evaluate  $\int_1^2 \frac{3x^2-7x+6}{(x-3)^2(x+1)} dx$ . Give the answer in the form of  $a + \ln b$ .

**SOLUTION**

$$\frac{3x^2 - 7x + 6}{(x - 3)^2(x + 1)} = \frac{A}{(x - 3)} + \frac{B}{(x - 3)^2} + \frac{C}{(x + 1)}$$

$$\frac{3x^2 - 7x + 6}{(x - 3)^2(x + 1)} = \frac{A(x - 3)(x + 1) + B(x + 1) + C(x - 3)^2}{(x - 3)^2(x + 1)}$$

$$3x^2 - 7x + 6 = A(x - 3)(x + 1) + B(x + 1) + C(x - 3)^2$$

When  $x = 3$ :

$$3(3)^2 - 7(3) + 6 = B(3 + 1)$$

$$27 - 21 + 6 = 4B$$

$$4B = 12$$

$$B = 3$$

When  $x = -1$ :

$$3(-1)^2 - 7(-1) + 6 = C[(-1) - 3]^2$$

$$16 = 16C$$

$$C = 1$$

When  $x = 0$ ,  $B=3$ ,  $C=1$ :

$$6 = A(-3)(1) + 3(1) + 1(-3)^2$$

$$6 = -3A + 3 + 9$$

$$3A = 6$$

$$A = 2$$

$$\frac{3x^2 - 7x + 6}{(x - 3)^2(x + 1)} = \frac{2}{(x - 3)} + \frac{3}{(x - 3)^2} + \frac{1}{(x + 1)}$$

$$\begin{aligned} \int \frac{3x^2 - 7x + 6}{(x - 3)^2(x + 1)} dx &= \int \frac{2}{(x - 3)} + \frac{3}{(x - 3)^2} + \frac{1}{(x + 1)} dx \\ &= \int \frac{2}{(x - 3)} dx + \int \frac{3}{(x - 3)^2} dx + \int \frac{1}{(x + 1)} dx \end{aligned}$$

$$\begin{aligned} &= 2 \int \frac{1}{(x-3)} dx + 3 \int (x-3)^{-2} dx + \int \frac{1}{(x+1)} dx \\ &= 2 \ln(x-3) + (3) \frac{(x-3)^{-2+1}}{-1} + \ln(x+1) \\ &= 2 \ln(x-3) - \frac{3}{(x-3)} + \ln(x+1) \\ &= \ln(x-3)^2 - \frac{3}{(x-3)} + \ln(x+1) \\ &= \ln[(x-3)^2(x+1)] - \frac{3}{(x-3)} \end{aligned}$$

$$\begin{aligned} \int_1^2 \frac{3x^2 - 7x + 6}{(x-3)^2(x+1)} dx &= \left[ \ln[(x-3)^2(x+1)] - \frac{3}{(x-3)} \right]_1^2 \\ &= \left[ \ln[(2-3)^2(2+1)] - \frac{3}{(2-3)} \right] - \left[ \ln[(1-3)^2(1+1)] - \frac{3}{(1-3)} \right] \\ &= \left[ \ln 3 - \frac{3}{(2-3)} \right] - \left[ \ln 8 - \frac{3}{(1-3)} \right] \\ &= \ln 3 - \ln 8 - \frac{3}{2} \\ &= \ln \left( \frac{3}{8} \right) + \frac{3}{2} \\ &= \frac{3}{2} + \ln \left( \frac{3}{8} \right) \end{aligned}$$



6. The Cartesian equation of an ellipse is given by  $4x^2 + 9y^2 - 16x + 36y + 16 = 0$ .
- Express the Cartesian equation of the ellipse in the standard form.
  - Determine the coordinates of the centre, vertices and foci of the ellipse.
  - Sketch the graph of the ellipse.

**SOLUTION**

(a)

$$4x^2 + 9y^2 - 16x + 36y + 16 = 0$$

$$[4x^2 - 16x] + [9y^2 + 36y] + 16 = 0$$

$$4[x^2 - 4x] + 9[y^2 + 4y] + 16 = 0$$

$$4\left[x^2 - 4x + \left(\frac{4}{2}\right)^2 - \left(\frac{4}{2}\right)^2\right] + 9\left[y^2 + 4y + \left(\frac{4}{2}\right)^2 - \left(\frac{4}{2}\right)^2\right] + 16 = 0$$

$$4[x^2 - 4x + 2^2 - 2^2] + 9[y^2 + 4y + 2^2 - 2^2] + 16 = 0$$

$$4[(x - 2)^2 - 4] + 9[(y + 2)^2 - 4] + 16 = 0$$

$$4(x - 2)^2 - 16 + 9(y + 2)^2 - 36 + 16 = 0$$

$$4(x - 2)^2 + 9(y + 2)^2 = 36$$

$$\frac{4(x - 2)^2}{36} + \frac{9(y + 2)^2}{36} = \frac{36}{36}$$

$$\frac{(x-2)^2}{9} + \frac{(y+2)^2}{4} = 1$$

(b)

$$\frac{(x - 2)^2}{9} + \frac{(y + 2)^2}{4} = 1$$

Compare to

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

$$\text{Centre}(h, k) = (2, -2)$$

$$c^2 = a^2 - b^2$$

$$c^2 = 9 - 4 = 5$$

$$c = \sqrt{5}$$

$$\text{Vertices} = (2 \pm 3, -2) = (5, -2); (-1, -2)$$

$$\text{Foci} = (2 \pm \sqrt{5}, -2) = (2 + \sqrt{5}, -2); (2 - \sqrt{5}, -2)$$

(c)

