



QS 015/2
Matriculation Programme
Examination
Semester I
Session 2013/2014

1. Express $\frac{x^2}{x^2+3x+2}$ in partial fractions form.
2. State the values of R and α such that $3\sin\theta + 6\cos\theta = R\sin(\theta + \alpha)$ where $R > 0$ and $0^\circ \leq \alpha < 90^\circ$. Hence, solve $3\sin\theta + 6\cos\theta = \sqrt{5}$ for $0^\circ \leq \theta < 180^\circ$.
3. (a) Find the value of m if $\lim_{x \rightarrow 0} \frac{mx + 3x^2}{4x - 8x^2} = 3$.
 (b) Evaluate $\lim_{x \rightarrow 0} \frac{\sqrt{3-x} - \sqrt{3}}{x}$.
4. (a) Find $\frac{dy}{dx}$ if $y = \operatorname{cosec}\{\sin[\ln(x+1)]\}$.
 (b) Obtain the second derivative of $y = \frac{\cos 3x}{e^{2x}}$ and express your answer in the simplest form.
5. A cubic polynomial $P(x)$ has remainders 3 and 1 when divided by $(x-1)$ and $(x-2)$, respectively.
 - (a) Let $Q(x)$ be a linear factor such that $P(x) = (x-1)(x-2)Q(x) + \alpha x + \beta$, where α and β are constants. Find the remainder when $P(x)$ is divided by $(x-1)(x-2)$.
 - (b) Use the values of α and β from part (a) to determine $Q(x)$ if the coefficient of x^3 for $P(x)$ is 1 and $P(3) = 7$. Hence, solve for x if $P(x) = 7 - 3x$.
6. (a) State the definition of the continuity of a function at a point. Hence, find the value of d such that

$$f(x) = \begin{cases} e^{3x+d}, & x \leq 0 \\ 3x + 5, & x > 0 \end{cases}$$

Is continuous at $x = 0$.

- (b) A function f is defined by

$$f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ k(x-1), & x > 1 \end{cases}$$

Determine the value(s) of k if f is:

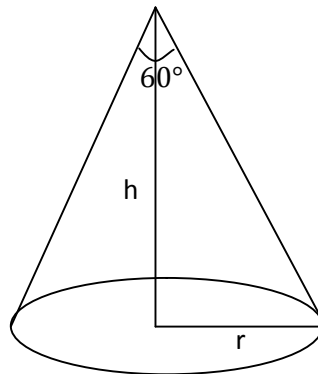
- (i) Continuous for all $x \in \mathbb{R}$.
- (ii) Differentiable for all $x \in \mathbb{R}$.

7. (a) Find the derivative of $f(x) = \frac{1}{x+1}$ by using the first principle.
- (b) Use implicit differentiation to find:
- (i) $\frac{dy}{dx}$ if $y \ln x = e^{x-y}$.
- (ii) the value of $\frac{dy}{dx}$ if $\frac{1}{y} - \frac{1}{x} = 3$ when $x = \frac{1}{2}$.

8. A curve is defined by parametric equations

$$x = \ln(1 + t), \quad y = e^{t^2} \text{ for } t > -1.$$

- (a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t .
- (b) Show that the curve has only one relative extremum at $(0,1)$ and determine the nature of the point.
9. (a) A cylindrical container of volume $128\pi m^3$ is to be constructed with the same material for the top, bottom and lateral side. Find the dimensions of the container that will minimise the amount of the material needed.
- (b) Gravel is poured onto a flat ground at the rate of $\frac{3}{20} m^3$ per minute to form a conical-shaped pile with vertex angle 60° as shown in the diagram below.



Compute the rate of change of the height of the conical pile at the instant $t = 10$ minutes.

10. (a) Show that $\frac{\sin \alpha + \sin \beta}{\cos \alpha - \cos \beta} = \cot \left[\frac{\beta - \alpha}{2} \right]$.
- (b) Use trigonometric identities to verify that

(i)
$$\sin \theta = \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$$

(ii)
$$\cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$$

Hence, solve the equation $3 \sin \theta + \cos \theta = 2$ for $0^\circ \leq \theta \leq 180^\circ$. Give your answers correct to three decimal places.

END OF QUESTION PAPER

1. Express $\frac{x^2}{x^2+3x+2}$ in partial fractions form.

SOLUTION

$$\begin{aligned} & x^2 + 3x + 2 \overline{) x^2 + 0x + 0} \\ & \underline{-x^2 - 3x - 2} \\ & 3x + 2 \end{aligned}$$

$$\frac{x^2}{x^2 + 3x + 2} = 1 - \frac{3x + 2}{x^2 + 3x + 2}$$

$$= 1 - \frac{3x + 2}{(x + 1)(x + 2)}$$

$$\frac{3x + 2}{(x + 1)(x + 2)} = \frac{A}{x + 1} + \frac{B}{x + 2}$$

$$= \frac{A(x + 2) + B(x + 1)}{(x + 1)(x + 2)}$$

$$3x + 2 = A(x + 2) + B(x + 1)$$

When $x = -1$

$$3(-1) + 2 = A(-1 + 2) + B(-1 + 1)$$

$$-1 = A(1)$$

$$A = -1$$

When $x = -2$

$$3(-2) + 2 = A(-2 + 2) + B(-2 + 1)$$

$$-4 = B(-1)$$

$$B = 4$$

$$\frac{3x + 2}{(x + 1)(x + 2)} = \frac{-1}{x + 1} + \frac{4}{x + 2}$$

$$\begin{aligned} \frac{x^2}{x^2 + 3x + 2} &= 1 - \frac{3x + 2}{(x + 1)(x + 2)} \\ &= 1 - \left[\frac{-1}{x + 1} + \frac{4}{x + 2} \right] \\ &= 1 + \frac{1}{x + 1} - \frac{4}{x + 2} \end{aligned}$$

2. State the values of R and α such that $3\sin\theta + 6\cos\theta = R\sin(\theta + \alpha)$ where $R > 0$ and $0^\circ \leq \alpha < 90^\circ$. Hence, solve $3\sin\theta + 6\cos\theta = \sqrt{5}$ for $0^\circ \leq \theta < 180^\circ$.

SOLUTION

$$3\sin\theta + 6\cos\theta = R\sin(\theta + \alpha)$$

$$\left. \begin{aligned} R &= \sqrt{3^2 + 6^2} = \sqrt{45} = 3\sqrt{5} \\ \alpha &= \tan^{-1} \frac{6}{3} = \tan^{-1} 2 = 63.4^\circ \end{aligned} \right\}$$

$$\begin{aligned} R &= \sqrt{a^2 + b^2} \\ \alpha &= \tan^{-1} \left(\frac{b}{a} \right) \end{aligned}$$

$$3\sin\theta + 6\cos\theta = \sqrt{5}$$

$$3\sqrt{5}\sin(\theta + 63.4^\circ) = \sqrt{5}$$

$$\sin(\theta + 63.4^\circ) = \frac{\sqrt{5}}{3\sqrt{5}}$$

$$\sin(\theta + 63.4^\circ) = \frac{1}{3}$$

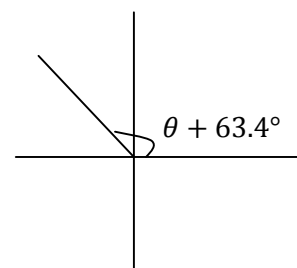
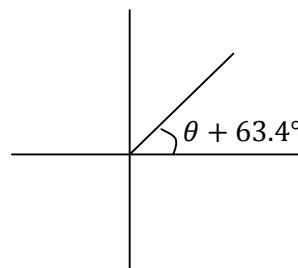
$$\theta + 63.4^\circ = \sin^{-1} \left(\frac{1}{3} \right) = 19.5^\circ, 160.5^\circ$$

$$\theta = 19.5^\circ - 63.4^\circ, \quad 160.5^\circ - 63.4^\circ$$

$$\theta = -43.9^\circ, \quad 97.1^\circ$$

Given that $0^\circ \leq \theta < 180^\circ$.

$$\therefore \theta = 97.1^\circ$$



3. (a) Find the value of m if $\lim_{x \rightarrow 0} \frac{mx + 3x^2}{4x - 8x^2} = 3$.
- (b) Evaluate $\lim_{x \rightarrow 0} \frac{\sqrt{3-x} - \sqrt{3}}{x}$.

SOLUTION**3(a)**

$$\lim_{x \rightarrow 0} \frac{mx + 3x^2}{4x - 8x^2} = 3$$

$$\lim_{x \rightarrow 0} \frac{x(m + 3x)}{x(4 - 8x)} = 3$$

$$\lim_{x \rightarrow 0} \frac{(m + 3x)}{(4 - 8x)} = 3$$

$$\frac{m + 3(0)}{4 - 8(0)} = 3$$

$$\frac{m}{4} = 3$$

$$m = 12$$

3(b)

$$\lim_{x \rightarrow 0} \frac{\sqrt{3-x} - \sqrt{3}}{x} = \lim_{x \rightarrow 0} \frac{\sqrt{3-x} - \sqrt{3}}{x} \times \frac{\sqrt{3-x} + \sqrt{3}}{\sqrt{3-x} + \sqrt{3}}$$

$$= \lim_{x \rightarrow 0} \frac{(3-x) - 3}{x(\sqrt{3-x} + \sqrt{3})}$$

$$= \lim_{x \rightarrow 0} \frac{-x}{x(\sqrt{3-x} + \sqrt{3})}$$

$$= \lim_{x \rightarrow 0} \frac{-1}{\sqrt{3-x} + \sqrt{3}}$$

$$= \frac{-1}{\sqrt{3-0} + \sqrt{3}}$$

$$= \frac{-1}{2\sqrt{3}}$$

4. (a) Find $\frac{dy}{dx}$ if $y = \operatorname{cosec}\{\sin[\ln(x+1)]\}$.
- (b) Obtain the second derivative of $y = \frac{\cos 3x}{e^{2x}}$ and express your answer in the simplest form.

SOLUTION**4(a)**

$$y = \operatorname{cosec}\{\sin[\ln(x+1)]\}$$

$$\frac{dy}{dx} = -\operatorname{cosec}\{\sin[\ln(x+1)]\} \cot\{\sin[\ln(x+1)]\} \frac{d}{dx}\{\sin[\ln(x+1)]\}$$

$$\frac{dy}{dx} = -\operatorname{cosec}\{\sin[\ln(x+1)]\} \cot\{\sin[\ln(x+1)]\} \cos[\ln(x+1)] \frac{d}{dx}[\ln(x+1)]$$

$$\frac{dy}{dx} = -\operatorname{cosec}\{\sin[\ln(x+1)]\} \cot\{\sin[\ln(x+1)]\} \cos[\ln(x+1)] \frac{1}{(x+1)} \frac{d}{dx}(x+1)$$

$$\frac{dy}{dx} = -\operatorname{cosec}\{\sin[\ln(x+1)]\} \cot\{\sin[\ln(x+1)]\} \cos[\ln(x+1)] \frac{1}{(x+1)} \quad (1)$$

$$\frac{dy}{dx} = \frac{-\operatorname{cosec}\{\sin[\ln(x+1)]\} \cot\{\sin[\ln(x+1)]\} \cos[\ln(x+1)]}{(x+1)}$$

4(b)

$$y = \frac{\cos 3x}{e^{2x}}$$

$$u = \cos 3x$$

$$v = e^{2x}$$

$$u' = -3\sin 3x$$

$$v' = 2e^{2x}$$

$$\frac{dy}{dx} = \frac{(e^{2x})(-3\sin 3x) - (\cos 3x)(2e^{2x})}{(e^{2x})^2}$$

$$\frac{dy}{dx} = \frac{(e^{2x})[-3\sin 3x - 2\cos 3x]}{(e^{2x})^2}$$

$$\frac{dy}{dx} = \frac{-3\sin 3x - 2\cos 3x}{e^{2x}}$$

$$u = -3\sin 3x - 2\cos 3x$$

$$v = e^{2x}$$

$$u' = -9\cos 3x + 6\sin 3x$$

$$v' = 2e^{2x}$$

$$\frac{d^2y}{dx^2} = \frac{(e^{2x})(-9\cos 3x + 6\sin 3x) - (-3\sin 3x - 2\cos 3x)(2e^{2x})}{(e^{2x})^2}$$

$$\frac{d^2y}{dx^2} = \frac{(e^{2x})[(-9\cos 3x + 6\sin 3x) - (2)(-3\sin 3x - 2\cos 3x)]}{(e^{2x})^2}$$

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$

$$\frac{vu' - uv'}{v^2}$$

$$\frac{d^2y}{dx^2} = \frac{(e^{2x})[-9\cos 3x + 6\sin 3x + 6\sin 3x + 4\cos 3x]}{(e^{2x})^2}$$

$$\frac{d^2y}{dx^2} = \frac{-9\cos 3x + 6\sin 3x + 6\sin 3x + 4\cos 3x}{e^{2x}}$$

$$\frac{d^2y}{dx^2} = \frac{12\sin 3x - 5\cos 3x}{e^{2x}}$$

5. A cubic polynomial $P(x)$ has remainders 3 and 1 when divided by $(x - 1)$ and $(x - 2)$, respectively.
- (a) Let $Q(x)$ be a linear factor such that $P(x) = (x - 1)(x - 2)Q(x) + \alpha x + \beta$, where α and β are constants. Find the remainder when $P(x)$ is divided by $(x - 1)(x - 2)$.
- (b) Use the values of α and β from part (a) to determine $Q(x)$ if the coefficient of x^3 for $P(x)$ is 1 and $P(3) = 7$. Hence, solve for x if $P(x) = 7 - 3x$.

SOLUTION**5(a)**

$$P(1) = 3$$

$$P(2) = 1$$

$$P(x) = (x - 1)(x - 2)Q(x) + \alpha x + \beta$$

$$P(1) = (1 - 1)(1 - 2)Q(1) + \alpha(1) + \beta = 3$$

$$\alpha + \beta = 3 \dots\dots\dots (1)$$

$$P(2) = (2 - 1)(2 - 2)Q(2) + \alpha(2) + \beta = 1$$

$$2\alpha + \beta = 1 \dots\dots\dots (2)$$

$$(1) - (2)$$

$$\alpha - 2\alpha = 3 - 1$$

$$-\alpha = 2$$

$$\alpha = -2$$

$$\beta = 5$$

$$P(x) = (x - 1)(x - 2)Q(x) - 2x + 5$$

\therefore The remainder when $P(x)$ is divided by $(x - 1)(x - 2)$ is $-2x + 5$

$$P(x) = D(x)Q(x) + R(x)$$

$$P(x) = \underbrace{(x - 1)(x - 2)}_{D(x)} Q(x) + \underbrace{-2x + 5}_{R(x)}$$

5(b)

$$P(x) = (x - 1)(x - 2)Q(x) - 2x + 5$$

Since the coefficient of x^3 for $P(x)$ is 1 $\rightarrow Q(x) = (x + c)$

$$P(x) = (x - 1)(x - 2)(x + c) - 2x + 5$$

Given that $P(3) = 7$

$$P(3) = (3 - 1)(3 - 2)(3 + c) - 2(3) + 5 = 7$$

$$6 + 2c = 8$$

$$c = 1$$

$$Q(x) = (x + 1)$$

$$P(x) = (x - 1)(x - 2)(x + 1) - 2x + 5$$

$$P(x) = 7 - 3x$$

$$(x - 1)(x - 2)(x + 1) - 2x + 5 = 7 - 3x$$

$$(x - 1)(x - 2)(x + 1) - 2x + 3x + 5 - 7 = 0$$

$$(x - 1)(x - 2)(x + 1) + x - 2 = 0$$

$$(x - 1)(x - 2)(x + 1) + (x - 2) = 0$$

$$(x - 2)[(x - 1)(x + 1) + 1] = 0$$

$$(x - 2)[x^2 + x - x - 1 + 1] = 0$$

$$(x - 2)[x^2] = 0$$

$$x = 2 \quad \text{or} \quad x = 0$$

6. (a) State the definition of the continuity of a function at a point. Hence, find the value of d such that

$$f(x) = \begin{cases} e^{3x+d}, & x \leq 0 \\ 3x + 5, & x > 0 \end{cases}$$

Is continuous at $x = 0$.

- (b) A function f is defined by

$$f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ k(x - 1), & x > 1 \end{cases}$$

Determine the value(s) of k if f is:

- (iii) Continuous for all $x \in \mathbb{R}$.
 (iv) Differentiable for all $x \in \mathbb{R}$.

SOLUTION

6(a)

Definition of continuity:

A function f is continuous at a point $x = c$ in the domain of f if the following three conditions are satisfied:

- i. $f(c)$ is defined
- ii. $\lim_{x \rightarrow c} f(x)$ exists or finite
- iii. $\lim_{x \rightarrow c} f(x) = f(c)$

For $f(x)$ to be continuous at $x = 0$,

- (i) $f(0) = e^{3(0)+d} = e^d$
- (ii) $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} 3x + 5 = 5$
- (iii) $e^d = 5$
 $d = \ln 5$

6(bi)

$$f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ k(x - 1), & x > 1 \end{cases}$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x)$$

$$\lim_{x \rightarrow 1^-} x^2 - 1 = \lim_{x \rightarrow 1^+} k(x - 1)$$

$$1^2 - 1 = k(1 - 1)$$

$$0 = 0$$

Thus, k can take any real value for the continuity: $k \in \mathbb{R}$

6(bii)

$$f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ k(x - 1), & x > 1 \end{cases}$$

$$f'(a) = \lim_{x \rightarrow 1^-} \frac{f(x) - f(a)}{x - a}$$

$$f'(1^-) = \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1}$$

$$f'(1^-) = \lim_{x \rightarrow 1^-} \frac{(x^2 - 1) - (1^2 - 1)}{x - 1}$$

$$f'(1^-) = \lim_{x \rightarrow 1^-} \frac{x^2 - 1}{x - 1}$$

$$f'(1^-) = \lim_{x \rightarrow 1^-} \frac{(x+1)(x-1)}{x-1}$$

$$f'(1^-) = \lim_{x \rightarrow 1^-} (x+1)$$

$$f'(1^-) = 1+1$$

$$f'(1^-) = 2$$

$$f'(1^+) = \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1}$$

$$f'(1^+) = \lim_{x \rightarrow 1^+} \frac{(k(x-1)) - (1^2 - 1)}{x - 1}$$

$$f'(1^+) = \lim_{x \rightarrow 1^+} \frac{k(x-1)}{x-1}$$

$$f'(1^+) = \lim_{x \rightarrow 1^+} k$$

$$f'(1^+) = k$$

For f to be differentiable at $x = 1$,

$$f'(1^+) = f'(1^-)$$

$$k = 2$$

7. (a) Find the derivative of $f(x) = \frac{1}{x+1}$ by using the first principle.
- (b) Use implicit differentiation to find:
- (i) $\frac{dy}{dx}$ if $y \ln x = e^{x-y}$.
- (ii) the value of $\frac{dy}{dx}$ if $\frac{1}{y} - \frac{1}{x} = 3$ when $x = \frac{1}{2}$.

SOLUTION**7(a)**

$$f(x) = \frac{1}{x+1}$$

$$f(x+h) = \frac{1}{(x+h)+1}$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{(x+h)+1} - \frac{1}{x+1} \right)$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1(x+1) - 1(x+h+1)}{(x+h+1)(x+1)} \right)$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{x+1-x-h-1}{(x+h+1)(x+1)} \right)$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-h}{(x+h+1)(x+1)} \right)$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \left(\frac{-1}{(x+h+1)(x+1)} \right)$$

$$\frac{dy}{dx} = \frac{-1}{(x+0+1)(x+1)}$$

$$\frac{dy}{dx} = \frac{-1}{(x+1)^2}$$

First principle

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{1}{h} (f(x+h) - f(x))$$

7(bi)

$$y \ln x = e^{x-y}$$

$$y \left(\frac{1}{x} \right) + \ln x \left(\frac{dy}{dx} \right) = e^{x-y} \frac{d}{dx} (x-y)$$

$$\frac{y}{x} + \ln x \left(\frac{dy}{dx} \right) = e^{x-y} \left[1 - \frac{dy}{dx} \right]$$

$$\frac{y}{x} + \ln x \left(\frac{dy}{dx} \right) = e^{x-y} - e^{x-y} \frac{dy}{dx}$$

$$\ln x \left(\frac{dy}{dx} \right) + e^{x-y} \frac{dy}{dx} = e^{x-y} - \frac{y}{x}$$

$$\left(\frac{dy}{dx} \right) [\ln x + e^{x-y}] = e^{x-y} - \frac{y}{x}$$

$$\left(\frac{dy}{dx} \right) = \frac{e^{x-y} - \frac{y}{x}}{\ln x + e^{x-y}}$$

7(bii)

$$\frac{1}{y} - \frac{1}{x} = 3$$

$$\text{when } x = \frac{1}{2}$$

$$\frac{1}{y} - \frac{1}{\left(\frac{1}{2}\right)} = 3$$

$$\frac{1}{y} - 2 = 3$$

$$\frac{1}{y} = 3 + 2$$

$$\frac{1}{y} = 5$$

$$y = \frac{1}{5}$$

$$\frac{1}{y} - \frac{1}{x} = 3$$

$$y^{-1} - x^{-1} = 3$$

$$(-1)y^{-2} \frac{dy}{dx} - (-1)x^{-2} = 0$$

$$\frac{-1}{y^2} \frac{dy}{dx} + \frac{1}{x^2} = 0$$

$$\frac{-1}{y^2} \frac{dy}{dx} = -\frac{1}{x^2}$$

$$\frac{dy}{dx} = -\frac{1}{x^2} \cdot \frac{y^2}{-1}$$

$$\frac{dy}{dx} = \frac{y^2}{x^2}$$

$$\text{When } x = \frac{1}{2}, \quad y = \frac{1}{5}$$

$$\frac{dy}{dx} = \frac{\left(\frac{1}{5}\right)^2}{\left(\frac{1}{2}\right)^2}$$

$$\frac{dy}{dx} = \frac{\frac{1}{25}}{\frac{1}{4}}$$

$$\frac{dy}{dx} = \frac{1}{25} \cdot \frac{4}{1}$$

$$\frac{dy}{dx} = \frac{4}{25}$$

8. A curve is defined by parametric equations

$$x = \ln(1 + t), \quad y = e^{t^2} \quad \text{for } t > -1.$$

- (c) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t .
 (d) Show that the curve has only one relative extremum at (0,1) and determine the nature of the point.

SOLUTION

8(a)

$$x = \ln(1 + t),$$

$$y = e^{t^2}$$

$$\frac{dx}{dt} = \frac{1}{1+t} \frac{d}{dt}(1+t)$$

$$\frac{dy}{dt} = e^{t^2} \frac{d}{dt}(t^2)$$

$$\frac{dx}{dt} = \frac{1}{1+t} (1)$$

$$\frac{dy}{dt} = e^{t^2} (2t)$$

$$\frac{dx}{dt} = \frac{1}{1+t}$$

$$\frac{dy}{dt} = 2te^{t^2}$$

$$\frac{dt}{dx} = 1 + t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = (2te^{t^2}) \cdot (1 + t)$$

$$\frac{dy}{dx} = 2te^{t^2}(1 + t)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt} \left[\frac{dy}{dx} \right] \cdot \left[\frac{dt}{dx} \right]$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt} \left[\frac{dy}{dx} \right] \cdot \left[\frac{dt}{dx} \right]$$

$$\frac{d^2y}{dx^2} = \left\{ \frac{d}{dt} [2te^{t^2}(1 + t)] \right\} \cdot [1 + t]$$

$$u = 2te^{t^2}$$

$$v = 1 + t$$

$$u' = (2t)(2te^{t^2}) + (e^{t^2})(2)$$

$$v' = 1$$

$$u' = 4t^2e^{t^2} + 2e^{t^2}$$

$$\frac{d^2y}{dx^2} = \{(2te^{t^2})(1) + (1 + t)(4t^2e^{t^2} + 2e^{t^2})\} \cdot [1 + t]$$

$$\frac{d^2y}{dx^2} = \{2te^{t^2} + 4t^2e^{t^2} + 2e^{t^2} + 4t^3e^{t^2} + 2te^{t^2}\} \cdot [1 + t]$$

$$\frac{d^2y}{dx^2} = \{2e^{t^2} + 4te^{t^2} + 4t^2e^{t^2} + 4t^3e^{t^2}\} \cdot [1 + t]$$

$$\frac{d^2y}{dx^2} = \{2e^{t^2}(1 + 2t + 2t^2 + 2t^3)\} \cdot [1 + t]$$

8(b)

For extremum point(s), let $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 2te^{t^2}(1 + t)$$

$$2te^{t^2}(1 + t) = 0$$

$$t = 0 \quad \text{or} \quad t = -1$$

Since given that $t > -1$, the curve has only one extremum point when $t = 0$.

When $t = 0$,

$$x = \ln(1 + t), \quad y = e^{t^2}$$

$$x = \ln(1 + 0), \quad y = e^{0^2}$$

$$x = \ln(1), \quad y = e^0$$

$$x = 0, \quad y = 1$$

$\Rightarrow (0, 1)$ is the extremum point.

At $t = 0$, the second derivative test gives

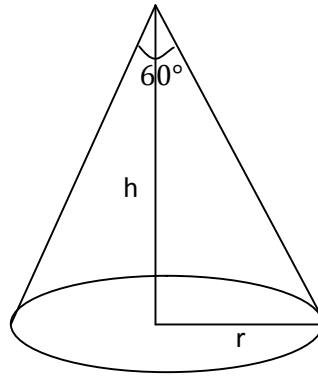
$$\frac{d^2y}{dx^2} = \{2e^{0^2}(1 + 2(0) + 2(0^2) + 2(0^3))\} \cdot [1 + (0)]$$

$$\frac{d^2y}{dx^2} = \{2(1)(1 + 0 + 0 + 0)\} \cdot [1 + (0)]$$

$$\frac{d^2y}{dx^2} = 2 > 0$$

Thus, the extremum point $(0,1)$ is a relative minimum point.

9. (a) A cylindrical container of volume $128\pi m^3$ is to be constructed with the same material for the top, bottom and lateral side. Find the dimensions of the container that will minimise the amount of the material needed.
- (b) Gravel is poured onto a flat ground at the rate of $\frac{3}{20} m^3$ per minute to form a conical-shaped pile with vertex angle 60° as shown in the diagram below.



Compute the rate of change of the height of the conical pile at the instant $t = 10$ minutes.

SOLUTION

$$V = \pi r^2 h = 128\pi$$

→ To minimise the **surface area**.

$$h = \frac{128\pi}{\pi r^2}$$

$$h = \frac{128}{r^2}$$

To minimise the **surface area**

$$S = 2\pi r^2 + 2\pi r h$$

$$S = 2\pi r^2 + 2\pi r \left(\frac{128}{r^2} \right)$$

$$S = 2\pi r^2 + \frac{256\pi}{r}$$

$$S = 2\pi r^2 + 256\pi r^{-1}$$

$$\frac{dS}{dr} = 4\pi r - 256\pi r^{-2}$$

$$\frac{dS}{dr} = 4\pi r - \frac{256\pi}{r^2}$$

$$\text{Let } \frac{dS}{dr} = 0$$

$$4\pi r - \frac{256\pi}{r^2} = 0$$

$$\frac{256\pi}{r^2} = 4\pi r$$

$$4\pi r^3 = 256\pi$$

$$r^3 = \frac{256\pi}{4\pi}$$

$$r^3 = \frac{256}{4}$$

$$r^3 = 64$$

$$r = 4$$

When $r = 4$,

$$h = \frac{128}{r^2}$$

$$h = \frac{128}{4^2}$$

$$h = 8$$

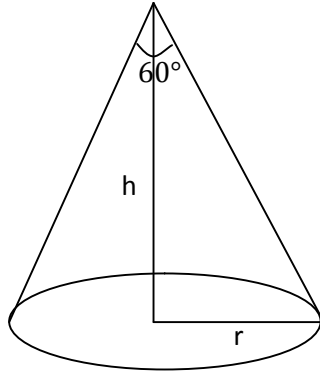
$$\frac{dS}{dr} = 4\pi r - \frac{256\pi}{r^2}$$

$$\frac{d^2S}{dr^2} = 4\pi + \frac{512\pi}{r^3}$$

When $r = 4$

$$\frac{d^2S}{dr^2} = 4\pi + \frac{512\pi}{4^3} = 12\pi > 0$$

$h = 8$ and $r = 4$ give minimum total surface area by the second derivative test

9(b)

$$\frac{dV}{dt} = \frac{3}{20} m^3 \quad \text{Find } \frac{dh}{dt} \text{ when } t = 10 \text{ minutes}$$

$$\tan 30^\circ = \frac{r}{h}$$

$$\frac{1}{\sqrt{3}} = \frac{r}{h}$$

$$\sqrt{3}r = h$$

$$r = \frac{h}{\sqrt{3}}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$$

$$V = \frac{1}{3} \pi r^2 h$$

$$V = \frac{1}{3} \pi \left(\frac{h}{\sqrt{3}} \right)^2 h$$

$$V = \frac{1}{3} \pi \left(\frac{h^2}{3} \right) h$$

$$V = \frac{\pi}{9} h^3$$

$$\frac{dV}{dh} = \frac{\pi}{3} h^2$$

$$\frac{dV}{dh} = \frac{\pi h^2}{3}$$

$$\frac{dh}{dV} = \frac{3}{\pi h^2}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$$

$$\frac{dh}{dt} = \frac{3}{\pi h^2} \cdot \frac{3}{20}$$

$$\frac{dh}{dt} = \frac{9}{20\pi h^2}$$

When $t = 10$,

$$\frac{dV}{dt} = \frac{3}{20} \rightarrow V = \frac{3}{20} t$$

$$V = \frac{3}{20} (10) = \frac{3}{2}$$

When $V = \frac{3}{2}$

$$\frac{3}{2} = \frac{\pi}{9} h^3$$

$$h^3 = \frac{9}{\pi} \cdot \frac{3}{2}$$

$$h^3 = \frac{27}{2\pi}$$

$$h = \left(\frac{27}{2\pi}\right)^{\frac{1}{3}}$$

$$\frac{dh}{dt} = \frac{9}{20\pi \left(\frac{27}{2\pi}\right)^{\frac{2}{3}}}$$

$$\frac{dh}{dt} = \frac{9}{20\pi} \left(\frac{2\pi}{27}\right)^{\frac{2}{3}} = 0.0542$$

10. (a) Show that $\frac{\sin \alpha + \sin \beta}{\cos \alpha - \cos \beta} = \cot \left[\frac{\beta - \alpha}{2} \right]$.
- (b) Use trigonometric identities to verify that

$$(i) \quad \sin \theta = \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$$

$$(iii) \quad \cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$$

Hence, solve the equation $3 \sin \theta + \cos \theta = 2$ for $0^\circ \leq \theta \leq 180^\circ$. Give your answers correct to three decimal places.

SOLUTION

10(a)

$$\begin{aligned} \frac{\sin \alpha + \sin \beta}{\cos \alpha - \cos \beta} &= \frac{2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)}{-2 \sin \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right)} \\ &= \frac{\cos \left(\frac{\alpha - \beta}{2} \right)}{-\sin \left(\frac{\alpha - \beta}{2} \right)} \\ &= -\cot \left(\frac{\alpha - \beta}{2} \right) \\ &= \cot \left(\frac{\beta - \alpha}{2} \right) \end{aligned}$$

10(bi)

$$\begin{aligned} \sin \theta &= 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \\ &= 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \left[\frac{\cos \frac{\theta}{2}}{\cos \frac{\theta}{2}} \right] \\ &= \frac{2 \sin \frac{\theta}{2} \cos^2 \frac{\theta}{2}}{\cos \frac{\theta}{2}} \\ &= 2 \tan \frac{\theta}{2} \cos^2 \frac{\theta}{2} \end{aligned}$$

$$\begin{aligned}
&= 2 \tan \frac{\theta}{2} \frac{1}{\frac{1}{\cos^2 \frac{\theta}{2}}} \\
&= 2 \tan \frac{\theta}{2} \frac{1}{\sec^2 \frac{\theta}{2}} \\
&= \frac{2 \tan \frac{\theta}{2}}{\sec^2 \frac{\theta}{2}} \\
&= \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}
\end{aligned}$$

10(bii)

$$\begin{aligned}
\cos \theta &= 1 - 2 \sin^2 \frac{\theta}{2} \\
&= \left(1 - 2 \sin^2 \frac{\theta}{2} \right) \cdot \left[\frac{\cos^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2}} \right] \\
&= \frac{\left(1 - 2 \sin^2 \frac{\theta}{2} \right) \cos^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2}} \\
&= \left(\frac{1}{\cos^2 \frac{\theta}{2}} - \frac{2 \sin^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2}} \right) \cdot \cos^2 \frac{\theta}{2} \\
&= \left(\sec^2 \frac{\theta}{2} - 2 \tan^2 \frac{\theta}{2} \right) \cdot \frac{1}{\frac{1}{\cos^2 \frac{\theta}{2}}} \\
&= \left(\sec^2 \frac{\theta}{2} - 2 \tan^2 \frac{\theta}{2} \right) \cdot \frac{1}{\sec^2 \frac{\theta}{2}} \\
&= \frac{\sec^2 \frac{\theta}{2} - 2 \tan^2 \frac{\theta}{2}}{\sec^2 \frac{\theta}{2}} \\
&= \frac{\left(1 + \tan^2 \frac{\theta}{2} \right) - 2 \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}
\end{aligned}$$

$$= \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$$

10(b)

$$3 \sin \theta + \cos \theta = 2$$

$$3 \left(\frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} \right) + \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} = 2$$

$$\text{Let } t = \tan \frac{\theta}{2}$$

$$3 \left(\frac{2t}{1 + t^2} \right) + \frac{1 - t^2}{1 + t^2} = 2$$

$$\frac{6t + 1 - t^2}{1 + t^2} = 2$$

$$6t + 1 - t^2 = 2(1 + t^2)$$

$$6t + 1 - t^2 = 2 + 2t^2$$

$$6t + 1 - t^2 = 2 + 2t^2$$

$$2 + 2t^2 + t^2 - 6t - 1 = 0$$

$$3t^2 - 6t + 1 = 0$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(1)}}{2(3)}$$

$$t = \frac{6 \pm \sqrt{36 - 12}}{6}$$

$$t = 0.1835$$

$$\text{or } t = 1.8165$$

$$\tan \frac{\theta}{2} = 0.1835$$

$$\tan \frac{\theta}{2} = 1.8165$$

$$\frac{\theta}{2} = \tan^{-1} 0.1835$$

$$\frac{\theta}{2} = \tan^{-1} 1.8165$$

$$\frac{\theta}{2} = 10.398^\circ$$

$$\frac{\theta}{2} = 61.167^\circ$$

$$\theta = 20.796^\circ$$

$$\theta = 122.334^\circ$$